Large-System Analysis of Static Multiuser Detection with an Unknown Number of Users

Adrià Tauste Campo
Universitat Pompeu Fabra, Barcelona, Spain
Email: adria.tauste@upf.edu

Ezio Biglieri
Universitat Pompeu Fabra, Barcelona, Spain
Email: ezio.biglieri@upf.edu

***Abstract*—We present a study of large multiple-access communication systems in which multiuser detection is performed without knowledge of the number of interferers. When the number of users increases without bound, optimum detectors can be analyzed asymptotically. A statistical physics approach based on spin glass theory provides analytical tools to deal with large systems in which the performance parameters to be analyzed (error probabilities, etc) are self-averaging in the limit. Of particular interest is the replica method that is used as a key technique to compute the free-energy function and the macroscopic parameters that determine the multiuser efficiency and the bit error probability in the large system limit.**

I. INTRODUCTION

In a multi-dimensional communications system many data streams are transmitted from various users to others via a common medium called a channel. In the case of a mobile multiple access communications scenario the number of active users, their location and other channel state parameters are variable with time.

Most research on multiuser detection theory is based on the assumption that the number of active users is constant and known at the receiver, and corresponds to the maximum number of users entitled to access the system. However, this model is quite pessimistic in a dynamic environment where there is a considerable number of users that remain inactive at any given time, and the assumption that the number of users is larger than the real leads to a performance loss for several families of detectors.

These dynamic conditions can be generalized to an environment where not only the number and the transmitted data, but also some continuous parameters of the active users are unknown at the receiver. Hence, we need to lay the foundations of multiuser detection theory for this scenario. For this purpose, two channels will be considered: static and dynamic channel. If there is an available dynamic model of the transmission system we will refer to the dynamic channel, otherwise, to the static channel.

This new perspective on multiuser communications requires a new approach from mathematics that considers the randomness not only in the data detected but also in the sets of users that are observed at any time. The underlying mathematical tool is called Random Set Theory (RST) and it can be used to derive new multiuser detectors (RST detectors) and improve the bit error probability (BEP) performance [1, 2].

A *random set* is a map from a sample space Ω to a family of subsets of a given space S. In the model we present, the space S is formed by the unknown data and parameters of the active interferers. Thus the information carried by the interferers at time t can be modeled as a random set:

$$\mathbf{X}_t = \{\mathbf{x}_t^{(1)}, \ldots, \mathbf{x}_t^{(k)}\} \tag{1}$$

whose elements (the parameters) are random vectors and k (the number of active interferers) itself is a random integer that takes values in the interval $[0, K-1]$, where K is the maximum number of users that can access the channel. If everything about the interferers is known except their number and identity, then all random sets $\mathbf{X}_t$ belong to the power set $2^{\{1,\ldots,K\}}$, that we will refer to as $2^{\mathbb{K}}$. Additionally, if we want to detect the (binary) data that the interferers transmit, $\mathbf{X}_t$ takes values in a set with 3^K elements, that we will denote as $3^{\mathbb{K}}$. Hence, the objective is to detect the number and identity of active interferers as well as the data they carry, assuming that the remaining parameters do not change during the tracking phase.

In a dynamic environment, the detection performance is determined by the channel and the dynamic model. The channel model is represented by the pdf $f_{\mathbf{z}}(\mathbf{y}_t|\mathbf{X}_t)$ where $\mathbf{y}_t$ is the singleton of the observed signal and $f_{\mathbf{z}}(.)$ is the probability density function of the additive noise. The dynamic model of the random sets sequence $\{\mathbf{X}_t\}_{t=1}^{\infty}$ is represented by the function $f_{\mathbf{X}_{t+1}|\mathbf{X}_t}(.|.)$ that describes the time evolution of data and parameters of the system under a Markovian assumption for the evolution of $\mathbf{X}_t$.

As an example of application of random sets to multiuser detection we examine here Direct Spectrum Code Division Multiple Access (DS-CDMA) systems where the received signal at time t is represented by:

$$\mathbf{y}_t = \mathbf{R}\mathbf{A}\mathbf{b}_{\mathrm{t}}(\mathbf{X}_{\mathrm{t}}) + \mathbf{z}_t, \quad t = 1, \ldots, T \tag{2}$$

where:

- $\mathbf{X}_t$ is the random set of active users.
- $\mathbf{R}$ is the correlation matrix of the signature sequences.
- $\mathbf{A}$ is the diagonal matrix of the users' signal amplitudes.
- The vector $\mathbf{b}_t(\mathbf{X}_t)$ has nonzero entries in the locations corresponding to the active-user identities described by $\mathbf{X}_t$
- $\mathbf{z}_t$ is the noise vector with Gaussian distribution expressed as $\mathcal{N}(0, (N_0/2)\mathbf{R})$ or $\mathcal{N}(0, \sigma^2\mathbf{R})$ with $N_0/2 = \sigma^2$ the power spectral density of the received noise.

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If we consider that example in the static channel, the a posteriori probability (APP) of $\mathbf{X}_t$, given the whole received sequence, is:

$$f(\mathbf{X}_t|\mathbf{y}_1, \ldots, \mathbf{y}_T) \tag{3}$$

Hence, a sequential maximum a posteriori probability (MAP) estimator of users' identities is:

$$\hat{\mathbf{X}} = \arg\max_{\mathbf{X}_t \in 2^{\mathcal{K}}} f(\mathbf{X}_t|\mathbf{y}_{1:T}) \tag{4}$$

In our analysis we will restrict ourselves to the case of the static channel and we will study the user and the user-and-data MAP detectors. In order to model both scenarios, suppose that the probability of interferer $x_t^{(i)}$ to be active is α, independent of t and i. Hence, the probability of interferer set $\mathbf{X}_t$ depends only of on its cardinality $|\mathbf{X}_t|$. Assume also that each user k transmits N antipodal binary information symbols (BPSK), which are independent from time to time and across users. Hence, the prior distributions in each detection case are:

$$f_{\mathbf{X}}(\mathbf{B}) = \alpha^{|\mathbf{B}|}(1-\alpha)^{K-|\mathbf{B}|} \tag{5}$$

$$f_{\mathbf{X}}(\mathbf{B}) = 2^{-N|\mathbf{B}|}\alpha^{|\mathbf{B}|}(1-\alpha)^{K-|\mathbf{B}|} \tag{6}$$

where the time subscripts are omitted for simplicity.

A. Statistical Physics Approach

In recent years, the theory of communications systems for multiple users have received important contributions from other disciplines. One of the most relevant contributions has come from statistical physics. Physicists have successfully built the theory of thermodynamics to explain the evolution of macroscopic values using a statistical description of the molecules. Using the same perspective, communications systems for multiple users can be modeled as well by statistical interactions between the signals belonging to different data streams with the purpose to derive properties in their large-system limit.

The relationship between communications and thermodynamics systems can be particularized in the concept of *free-energy*. In statistical physics, the free-energy $F(\mathbf{X})$ (where $\mathbf{X}$ is the state variable) relates the energy $E(\mathbf{X})$ and the entropy $H(\mathbf{X})$ of a physical system in the following way:

$$F(\mathbf{X}) = E(\mathbf{X}) - TH(\mathbf{X}) \tag{7}$$

where T is the temperature of the system. At thermal equilibrium the free-energy (7) is minimized, since the entropy is maximized as time evolves following the second law of thermodynamics. A key point here is that the free-energy normalized to the dimension of the system is a self-averaging quantity. Thus, the free-energy turns out to be the starting point for calculating macroscopic properties of a thermodynamic system.

The entropy plays a central role in information theory and then the relation established in (7) implies that the free-energy can be regarded from an information theoretic point of view. Actually, the only condition required to use the free energy is the existence of macroscopic variables, microscopic random variables and an energy function. In fact, for communications systems in practice the condition is that their size grows above all bounds since some (microscopic) user random variables (multiuser efficiency, bit error probability, etc..) are assumed to self-average in the large-system limit yielding macroscopic variables. In addition, the energy function can be interpreted as the metric of a detector in multiuser communications. From this perspective, any detector parameterized with a certain metric can be analyzed with the tools of statistical mechanics to derive results in the large system limit. Then, the calculation of the free-energy can be associated to parameters of a multiuser detection performance such as spectral efficiency, channel capacity and bit error probability, in large systems.

At thermodynamic equilibrium, the free-energy can also be expressed as $F = -T \log Z$ where $Z = \sum_x \exp\left(-\frac{1}{T}||\mathbf{x}||\right)$ is called the partition function, $||.||$ is the energy operator and the temperature $T \geq 0$ is determined by the energy constraint.

In a communications system, the detector has to infer the information-bearing symbols given the received signal $\mathbf{y}$ and knowledge about the channel state (parameterized by $\mathbf{S}$) which yields $Z(\mathbf{y}, \mathbf{S})$.

Thus, the free-energy of the thermodynamic system normalized by the number of users is ($T = 1$):

$$F_K = -1/K \log Z(\mathbf{y}, \mathbf{S}) \tag{8}$$

In order to calculate the free-energy we use the self-averaging assumption, which states that the randomness of (8) vanishes as $K \to \infty$. Hence, the free-energy per user converges in probability to its expected value over the distribution of the random variables $\mathbf{y}$ and $\mathbf{S}$:

$$F = -\lim_{K\to\infty} \mathbb{E}\{-1/K \log Z\} \tag{9}$$

However, there is a practical problem in the applicability of the statistical mechanics framework to the communications large-system analysis. The analytical calculations required to solve the equations arising from (9) may not be feasible.

The most commonly used tool to evaluate the free-energy was first developed for the analysis of some particular magnetic materials called spin glasses and is known as the *replica method*. Although lacking of strong mathematical basis in some respects, it has been shown to be consistent with engineering results from random matrix and free-probability theory. However, the most interesting point is that it allows us to make reasonable predictions in other engineering applications such as the dynamic multiuser detection described above. In that case, the calculation of the free-energy is useful to analyze RST detectors and to derive its BEP for a large number of users.

II. Replica Method Development and Presentation of Main Results

We generalize the classical statistical approach to randomly spread CDMA systems [3, 4, 5] according to the framework provided in [2] in order to define more realistic and powerful detectors based on MAP estimation.

For this purpose, we analyze the DS-CDMA system (2) with the following assumptions:

- $\mathbf{S} = \mathbf{RA}$ where $\mathbf{S}$ is an $N \times K$ matrix whose columns are the spreading sequences of the users and N is the spreading gain, or sequence length.
- For simplicity, we model the noise as $\mathbf{z} \sim \mathcal{N}(0, \sigma^2 \mathbf{I})$.

We also denote the asymptotic factor or system load as $\beta = K/N$, where K is the maximum number of active users at any given time.

Thus, the static DS-CDMA system is modeled as follows:

$$\mathbf{y} = \mathbf{S}\mathbf{b}(\mathbf{X}) + \mathbf{z} \tag{10}$$

Our approach is aimed at developing the free-energy function calculation (9) for (10) to obtain the macroscopic parameters that determine the bit error probability in the large system limit. The replica method, which we use to calculate the free-energy ($\mathcal{F}$), consists of computing $\mathcal{F}$ as follows:

$$\mathcal{F} = \lim_{n \to 0} \frac{\partial}{\partial_n} \left(\lim_{K \to \infty} \frac{1}{K} \log \mathbb{E}\{Z^n(\mathbf{y}, \mathbf{S})\} \right) \tag{11}$$

According to the statistical physics approach described above, we present our main results in the following two theorems:

Theorem 2.1: Given a randomly spread DS-CDMA communication scheme with constant equal power per user, the large-system multiuser efficiency η of a RST individually optimum detector that performs MAP estimation of user identities is the solution of the following fixed-point equation:

$$\eta = \frac{1}{1 + \beta \left(\gamma \left[\alpha - \int \sqrt{\frac{\eta}{2\pi}} \frac{\alpha^2 e^{-\frac{\eta}{2}(y - \sqrt{\gamma}))^2}}{\alpha + (1-\alpha) e^{-\eta(\sqrt{\gamma} y - \frac{\gamma}{2})}} \mathrm{d}y \right] \right)} \tag{12}$$

that minimizes the free-energy:

$$\mathcal{F} = -\mathbb{E}\left[\int (\phi \log \phi) \mathrm{d}y \right] - \frac{1}{2} \log \frac{2\pi e}{\eta} + \frac{1}{2\beta} \left(\eta \log e + \log \frac{2\pi e}{\eta} \right) \tag{13}$$

where α is the activity rate of users, β is the system load, γ is the signal to noise ratio (SNR) per user, $\mathbb{E}$ performs over all random variables and $\phi = \phi(y|\gamma, \eta, b(\mathbf{X}))$ is the large-system equivalent single-user Gaussian channel [3].

Hence, the large-system bit error probability P_b per user is computed as $P_b = Q\left(\sqrt{\eta\gamma}\right)$, where $Q(x) = \int_x^\infty \frac{1}{2\pi} e^{-t^2} \mathrm{d}t$

Theorem 2.2: Given a randomly spread DS-CDMA communication scheme with constant equal power per user, the large-system multiuser efficiency η of a RST individually optimum detector that performs MAP estimation of user identities and their data is the solution of the following fixed-point equation:

$$\eta = \frac{1}{1 + \beta \left(\gamma \left[\alpha - \int \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} \frac{\alpha^2 \sinh[\eta\gamma - y\sqrt{\eta\gamma}]}{\alpha \cosh[\eta\gamma - y\sqrt{\eta\gamma}] + (1-\alpha) e^{\eta \frac{\gamma}{2}}} \mathrm{d}y \right] \right)} \tag{14}$$

that minimizes the free-energy function (13). Again, the large-system bit error probability P_b per user is computed as $P_b = Q\left(\sqrt{\eta\gamma}\right)$.

Unlike other approaches [3, 4, 5], the fixed-point equations (12) and (14) are derived according to the prior distributions of random sets given in (5) and (6) that consider the users' activity in a static channel. Hence, under MAP estimation, the detection requires the knowledge not only of the prior information of the data, but also of the activity rate parameter α. Then, the fixed-point equations are determined by the minimum mean square error $\mathbb{E}_\gamma \left\{ \alpha - \int \frac{\mathbb{E}^2_{b(\mathbf{X})}\{b(\mathbf{X}) P(y|\eta, b(\mathbf{X}), S)\}}{\mathbb{E}_{b(\mathbf{X})}\{P(y|\eta, b(\mathbf{X}), S)\}} \mathrm{d}y \right\}$. The application of this expression for each prior distribution leads to the results shown above.

III. Numerical Results

Here, we show the asymptotic performance curves of a individually optimum RST detector that knows the user's activity rate α. In this context, we provide the large-system performance of this detector in two different types of random set estimation: user identification, where the a priori distribution of the random set is given by (5) and data detection, where it is given by (6).

Fig. 1 plots the large-system multiuser efficiency performance for MAP user identification based on the knowledge of α alone. The values of the multiuser efficiency are obtained numerically through the resolution of the fixed-point equation (13) for a system load $\beta = 1$. Fig. 2 plots the corresponding user bit error probability in a trained acquisition phase.

Fig. 3 plots the multiuser efficiency performance for a MAP detector of users and data based on the knowledge of α alone and assuming BPSK modulation. In this case, the values of the multiuser efficiency are obtained numerically through the resolution of the fixed-point equation (14) for a system load $\beta = 3/7$. Fig. 4 plots the corresponding user bit error probability for some values of α. The values obtained in the asymptotic regime match those obtained by simulation in [2], showing that our asymptotic analysis yields results that are also valid for a finite (and fairly small) number of users.

It is interesting to observe the performance of the multiuser efficiency for different values of α. All curves achieve a minimum value at a different SNR value. For instance, in the user-and-data detector the minimum is obtained for a value near $\alpha = 0.5$, which corresponds approximately to the maximum uncertainty under MAP estimation. This value also represents a threshold in the curve behavior of the multiuser efficiency. For values of α approaching 1 the shape of the curve tends to that of $\alpha = 1$, a fact also observed in [5]. For smaller values of α, the minimum increases its value and approaches the analytical limit 1 when $\alpha \to 0$. The existence of the minimum is confirmed by results previously reported in the literature [7, 6]. Firstly, the definition of multiuser efficiency [7] entails that at very low SNR, it tends to 1. Additionally, as proved in [6] the multiuser efficiency for binary antipodal transmission converges to 1 as SNR approaches infinity.

IV. Conclusions

We have developed a replica-method analysis from statistical physics for the class of multiuser RST detectors in an static environment. We have derived the large system multiuser efficiency and the user bit-error probability for the cases of user identification and data detection. The analytical results

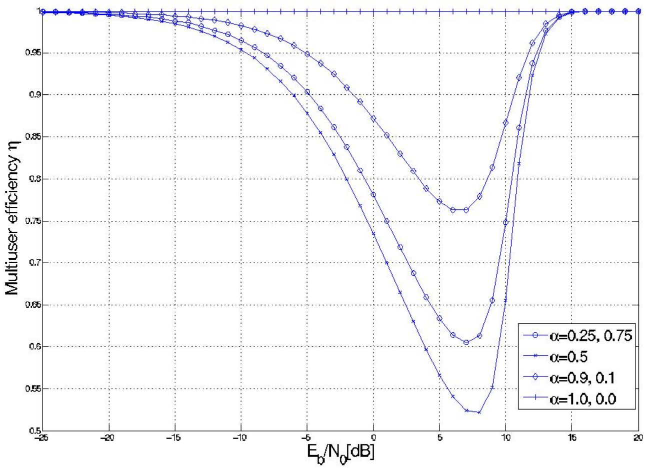

Figure 1. Large system multiuser efficiency of user identification under MAP detection with prior knowledge of α and β=1.

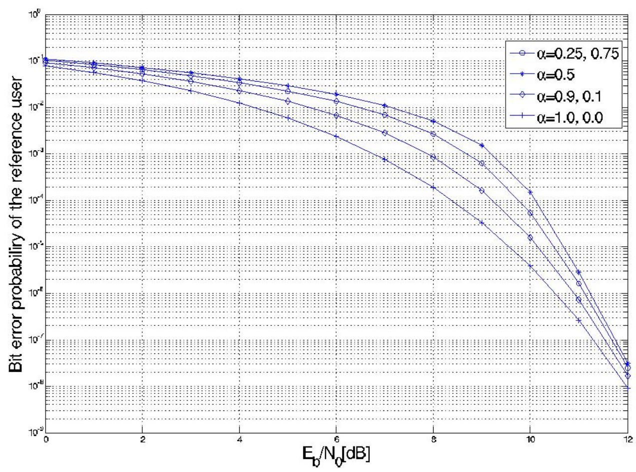

Figure 2. Large-system user bit error probability of user identification under MAP detection with prior knowledge of α and β=1.

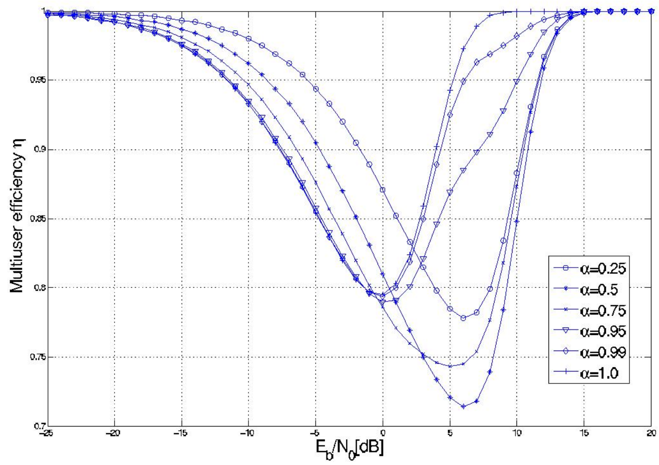

Figure 3. Large-system multiuser efficiency of user identification and data detection under MAP detection with prior knowledge of α and β=3/7.

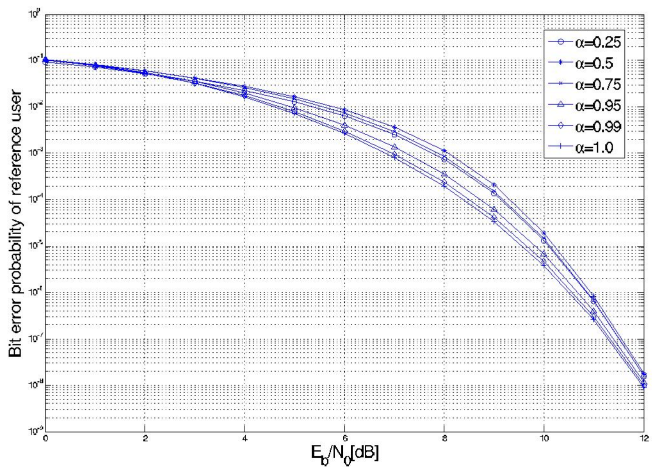

Figure 4. Large-system user bit error probability of user identification and data detection under MAP detection with prior knowledge of α and β=3/7.

match the simulations carried out in the seminal paper on RST detectors [2]. Moreover, if our results are specialized to the common assumption on the number of users ($\alpha = 1$), they coincide with previous results reported in the literature. For the purpose of the free-energy computation, we have provided a more general framework for time-varying multiuser schemes based on prior distributions of random sets. This can have additional interest when analyzing the dynamic channel given in [2] in the large-system limit.

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