Deriving the Global Game Selection in Games with many Actions and Asymmetric Players

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Global-Game Approach

Suppose, θ is uncertain

Let $\theta \sim \mathrm{N}\,(\mu,\tau^2)$

Players get private signals

$x^i = \theta + \varepsilon^i, \quad \varepsilon^i \sim \mathrm{N}(0,\sigma^2)$

ε^1 and ε^2 are independent

If σ/τ is sufficiently small, there is a unique equilibrium (for given μ) with a threshold signal x^*, s.t. agent i invests if $x^i > x^*$ and does not invest, if $x^i < x^*$.

Carlsson/van Damme (1993)

○

	invest	Not invest
invest	θ θ	0 $\theta- 1$
not invest	θ -1 0	0 0

Equilibrium condition:

$$E\left(\theta \middle| x^i = x^*\right)$$

$$= prob(x^j < x^* \mid x^i = x^*)$$

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In 2x2 games GGS=RDE

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Equilibrium condition:

$$E\left(\theta \middle| x^i = x^*\right)$$

$$= prob(x^j < x^* \mid x^i = x^*)$$

For $\sigma^2 \rightarrow 0$, the equilibrium converges to $x^* = 1/2$

"global-game selection"

Supermodular Game

Generalization:

Set of players I, ordered finite action sets $A_i = \{0,1,2, \dots ,m_i \}, i \in I$.

Actions $a_i \in A_i$, action profile $a \in A = \Pi_i A_i$,

lowest and highest action profiles 0 and m.

Complete information game Γ, specified by payoff functions $g_i : A \to \mathbf{R}$.

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Complete information game Γ, specified by payoff functions $g_i : A \rightarrow \mathbf{R}$.

- Game Γ is **supermodular** [actions are strategic complements], if for all i and for all $a_i \le a'_i$ and $a_{-i} \le a'_{-i}$:

$$g_i(a'_i , a_{-i}) - g_i(a_i, a_{-i}) \le g_i(a'_i , a'_{-i}) - g_i(a_i, a'_{-i}).$$

$\Rightarrow$ Best response functions are non-decreasing.

Supermodular games often have multiple equilibria.

Global Game – general definition

A **global game** G is defined by

- payoff functions $u_i(a_i, a_{-i}, \theta)$, where $\theta \in \mathbf{R}$ is called state parameter,

  s.t. (A1) $u_i(\cdot,\theta)$ is a <u>supermodular game</u>,

  (A2) $\exists \underline{\theta}$ and $\overline{\theta}$, s.t. the lowest and highest action are <u>strictly dominant</u> in the games given by $u_i(\cdot, \underline{\theta})$ and $u_i(\cdot, \overline{\theta})$,

  (A3) each u_i satisfies weak <u>state monotonicity</u>: for all i and $a_i < a'_i$:

  $u_i(a'_i, a_{-i}, \theta) - u_i(a_i, a_{-i}, \theta)$ is weakly increasing in θ.

  => higher states make higher actions more appealing.

○

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  => higher states make higher actions more appealing.

- a distribution for the state parameter with continuous density ϕ, and

- a tuple of density functions f_i with finite support for private signals η_i :

  In the global game, players do not observe state θ. They receive private signals $x_i = \theta + \nu\, \eta_i$ where $\nu \in (0,1]$ is a scale parameter.

Global Game – general definition

A **global game** G is defined by $(u,\ \phi,\ f,\ \nu)$

- payoff functions $u_i(a_i, a_{-i}, \theta)$, where $\theta \in \mathbf{R}$ is called state parameter,

  s.t. (A1) $u_i(\cdot, \theta)$ is a supermodular game,

  (A2) $\exists \underline{\theta}$ and $\overline{\theta}$, s.t. the lowest and highest action are strictly dominant in the games given by $u_i(\cdot, \underline{\theta})$ and $u_i(\cdot, \overline{\theta})$,

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Global Game selection

A global game G embeds a complete information game Γ at state θ^*, if

$$g_i(a) = u_i(a, \theta^*) \text{ for all } i, a.$$

Theorem (analogue to Frankel, Morris, and Pauzner, JET 2003):

As the scale parameter v goes to zero, the global game $G^v(u,\phi,f)$ has an essentially **unique limit equilibrium**.

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○ More precisely: denote a pure strategy of the GG by $s_i : \mathbf{R} \rightarrow A_i$, s.t. player i chooses action $s_i(x_i)$ when receiving signal x_i.

There is a strategy s, s.t. for $v \rightarrow 0$, any equilibrium $s^v(x)$ of $G^v(\cdot)$ converges to $s(x)$ for all x except possibly at the finitely many discontinuities of s.

If the global game‘s limit equilibrium strategy profile is continuous at θ^*, its value at that state determines a particular Nash equilibrium of the complete information game, called **global-game selection** (GGS).

Noise independence

Research Question:

Under which conditions is the GGS independent from u, ϕ, and f ?

FMP (2003) show general independence from ϕ.

BDH (2013) show general independence from u.

=> One may use w/o.l.o.g. $u_i(a,\theta) = g_i(a) + \theta\, a_i$.

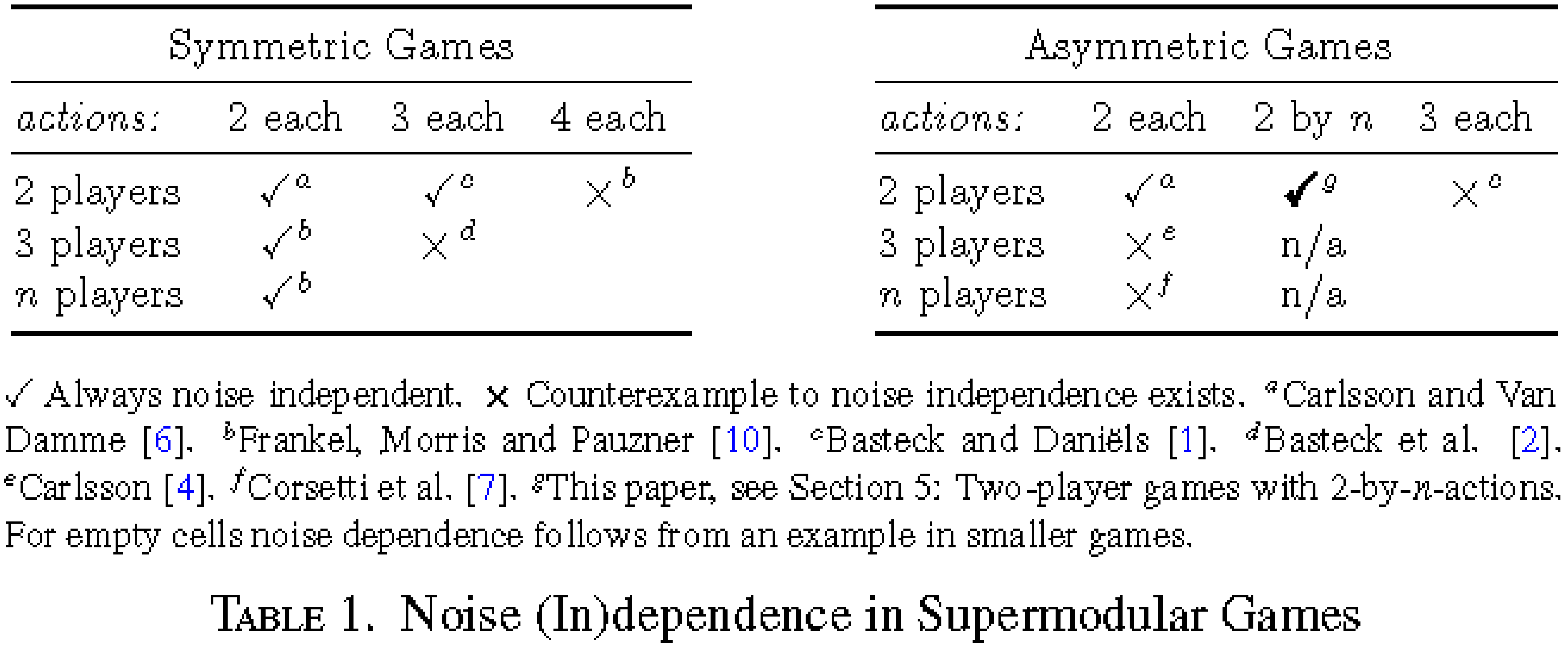
Proof: For a sufficiently wide support of ϕ, u_i satisfies the global-game assumptions (A1) to (A3). Obviously, u_i embeds g at $\theta^*=0$.

In general, the GGS may depend on f.

The GGS is called **noise independent**, if the GGS is independent of the particular densitiy function of private signals f.

Noise independence in Small Games

For certain small games, the GGS is known to be noise independent:



Symmetric Games			
actions:	2 each	3 each	4 each
2 players	✓[a]	✓[c]	×[b]
3 players	✓[b]	×[d]	
n players	✓[b]		

Asymmetric Games			
actions:	2 each	2 by n	3 each
2 players	✓[a]	✔[g]	×[c]
3 players	×[e]	n/a	
n players	×[f]	n/a	

✓ Always noise independent. × Counterexample to noise independence exists. [a]Carlsson and Van Damme [6]. [b]Frankel, Morris and Pauzner [10]. [c]Basteck and Daniëls [1]. [d]Basteck et al. [2]. [e]Carlsson [4]. [f]Corsetti et al. [7]. [g]This paper, see Section 5: Two-player games with 2-by-n-actions. For empty cells noise dependence follows from an example in smaller games.

TABLE 1. Noise (In)dependence in Supermodular Games

Decomposition Result

Definition: Consider a supermodular complete information game Γ with joint action set A. For action profiles $a \leq a'$, we define

$$[a, a'] = \{\tilde{a} \in A \,|\, a \leq \tilde{a} \leq a'\}.$$

The restricted game $\Gamma|[a,a']$ is given by restricting the joint action set of Γ.

Lemma (BDH, 2013): Consider a supermodular game Γ and a noise structure f. An action profile a^n is the unique GGS of Γ, if there is a sequence $0 = a^0 \leq a^1 \leq \ldots \leq a^n \leq \ldots \leq a^m = m$ s.t.

(i) a^j is the unique GGS in $\Gamma|[a^{j-1}, a^j]$ for all $j \leq n$, and

(ii) a^{j-1} is the unique GGS in $\Gamma|[a^{j-1}, a^j]$ for all $j > n$.

Corollary: If all the restricted games are noise independent, then Γ is also noise independent and a^n is the unique noise independent GGS of Γ.

Decomposition Result

Procedure

- Check supermodularity of the game.
- Decompose total game $[0,m]$ into a sequence of smaller games: $0 = a^0 \leq a^1 \leq \ldots \leq a^n \leq \ldots \leq a^m = m$
- Derive GGS of smaller games.
- If all solutions point to the same strategy, this is a GGS of the large game.

$$0 = a^0 \rightarrow a^1 \rightarrow a^2 \boxed{\rightarrow a^3 \leftarrow} a^4 \leftarrow a^5 = m$$

- If, in addition, all small games are noise independent, the large game is also noise independent.

Application 3: Asymmetric Players

Aquisition of a network good. Payoff depends on number of adopters.

V(i, n)	number of adopters n											
Player i	1	2	3	4	5	6	7	8	9	10	11	12
A	-4	-1	2	5	8	11	14	17	20	23	26	29
B	-4	-1	2	5	8	11	14	17	20	23	26	29
C	-4	-1	2	5	8	11	14	17	20	23	26	29
D	-13	-10	-7	-4	-1	2	5	8	11	14	17	20
E	-13	-10	-7	-4	-1	2	5	8	11	14	17	20
F	-13	-10	-7	-4	-1	2	5	8	11	14	17	20
G	-25	-22	-19	-13	-10	-7	-4	-1	2	5	8	11
H	-25	-22	-19	-13	-10	-7	-4	-1	2	5	8	11
I	-25	-22	-19	-13	-10	-7	-4	-1	2	5	8	11
J	-34	-31	-28	-25	-22	-19	-13	-10	-7	-4	-1	2
K	-34	-31	-28	-25	-22	-19	-13	-10	-7	-4	-1	2
L	-34	-31	-28	-25	-22	-19	-13	-10	-7	-4	-1	2

Application 3: Asymmetric Players

Aquisition of a network good. There are M types of players with different payoff functions.

$v_i(n)$ = agent i‘s payoff from entry if n players enter in total.

Agents with the same payoff function belong to the same type.

Order the types s.t. „i belongs to a lower type than j“ iff $v_i(n) \geq v_j(n)$ for all n with at least one strict inequality.

Strategy <u>combinations</u> are partially ordered: $a \geq a'$ iff $a_i \geq a'_i$ for all i.

Define a^0 as the strategy combination, where e.b. stays out,

a^1 as the strategy combination, where all players of type 1 enter, others stay out,

a^k as the strategy combination, where all players of types 1 to k enter and players of higher types stay out. => a^M = all players enter.

Application 3: Asymmetric Players

Look at restricted games with all strategies in $[a^{k-1}, a^k]$ for $k = 1, \ldots, M$.

In each of these games, only players of type k have to decide. It is described by payoffs on the block diagonal.

It is a symmetric binary-action game between players of the same type.

It is noise independent and the GGS is given by the best response of a type-k player to a uniform distribution on the number of entrants among the other players of his own type (with players of lower types entering, higher types staying out).

V(i, n)	number of adopters n		
Player i	1	2	3
A	-4	-1	2
B	-4	-1	2
C	-4	-1	2

$\Rightarrow$ Expected payoff = -1

$\Rightarrow$ a^{k-1} is selected

$\Rightarrow$ GGS: no player enters.

Application 3: Asymmetric Players

Experiment: 4 sessions with 12 players each.

Subjects were playing 20 different games in random order without feedback. Roles were randomly assigned to subjects for each game independently.

In each game, subjects could choose between two options:

- For option A, they received 34 ECU.
- Payoffs for option B were presented by payoff tables.

First, subjects had to answer comprehensive questions to make sure that they understood how to read the payoff tables.

Alternative solutions:

- Naive global game: best response to random number of entries
- Levels of reasoning: Level 1: best response to each subject enetering with probability of 50%, Level k = best response to Level k – 1.

Application 3: Asymmetric Players

Game 1 Naive GGS predicts entry of A-C, Level k: A-C

V(i, n)	number of adopters n												observed
Player i	1	2	3	4	5	6	7	8	9	10	11	12	entries
A	30	33	36	36	36	36	36	36	36	36	36	36	9 / 12
B	30	33	36	36	36	36	36	36	36	36	36	36	
C	30	33	36	36	36	36	36	36	36	36	36	36	
D	0	0	0	30	33	36	36	36	36	36	36	36	1 / 12
E	0	0	0	30	33	36	36	36	36	36	36	36	
F	0	0	0	30	33	36	36	36	36	36	36	36	
G	0	0	0	0	0	0	30	33	36	36	36	36	0
H	0	0	0	0	0	0	30	33	36	36	36	36	
I	0	0	0	0	0	0	30	33	36	36	36	36	
J	0	0	0	0	0	0	0	0	0	30	33	36	0
K	0	0	0	0	0	0	0	0	0	30	33	36	
L	0	0	0	0	0	0	0	0	0	30	33	36	
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Application 3: Asymmetric Players

Game 2 (higher payoffs above diagonal compared to Game 1)

Naive GGS predicts entry of A-C, Level k: A-F

V(i, n)	number of adopters n												observed
Player i	1	2	3	4	5	6	7	8	9	10	11	12	entries
A	30	33	36	39	42	45	48	51	54	57	60	63	12 / 12
B	30	33	36	39	42	45	48	51	54	57	60	63	
C	30	33	36	39	42	45	48	51	54	57	60	63	
D	0	0	0	30	33	36	39	42	45	48	51	54	4 / 12
E	0	0	0	30	33	36	39	42	45	48	51	54	
F	0	0	0	30	33	36	39	42	45	48	51	54	
G	0	0	0	0	0	0	30	33	36	39	42	45	1 / 12
H	0	0	0	0	0	0	30	33	36	39	42	45	
I	0	0	0	0	0	0	30	33	36	39	42	45	
J	0	0	0	0	0	0	0	0	0	30	33	36	0
K	0	0	0	0	0	0	0	0	0	30	33	36	
L	0	0	0	0	0	0	0	0	0	30	33	36	
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Application 3: Asymmetric Players

Game 3 (higher payoffs below the diagonal compared to Game 1)

Naive GGS predicts entry of A-C, Level k: A-F

V(i, n)	number of adopters n												observed
Player i	1	2	3	4	5	6	7	8	9	10	11	12	entries
A	30	33	36	36	36	36	36	36	36	36	36	36	10 / 12
B	30	33	36	36	36	36	36	36	36	36	36	36	
C	30	33	36	36	36	36	36	36	36	36	36	36	
D	21	24	27	30	33	36	36	36	36	36	36	36	5 / 12
E	21	24	27	30	33	36	36	36	36	36	36	36	
F	21	24	27	30	33	36	36	36	36	36	36	36	
G	9	12	15	21	24	27	30	33	36	36	36	36	0
H	9	12	15	21	24	27	30	33	36	36	36	36	
I	9	12	15	21	24	27	30	33	36	36	36	36	
J	0	3	6	9	12	15	21	24	27	30	33	36	0
K	0	3	6	9	12	15	21	24	27	30	33	36	
L	0	3	6	9	12	15	21	24	27	30	33	36	
													15 / 48

Application 3: Asymmetric Players

Game 4 (higher payoffs above and below the diagonal cf. Game 1)

Naive GGS predicts entry of A-F, Level k: A-F

V(i, n)	number of adopters n												observed
Player i	1	2	3	4	5	6	7	8	9	10	11	12	entries
A	30	33	36	39	42	45	48	51	54	57	60	63	11 / 12
B	30	33	36	39	42	45	48	51	54	57	60	63	
C	30	33	36	39	42	45	48	51	54	57	60	63	
D	21	24	27	30	33	36	39	42	45	48	51	54	7 / 12
E	21	24	27	30	33	36	39	42	45	48	51	54	
F	21	24	27	30	33	36	39	42	45	48	51	54	
G	9	12	15	21	24	27	30	33	36	39	42	45	2 / 12
H	9	12	15	21	24	27	30	33	36	39	42	45	
I	9	12	15	21	24	27	30	33	36	39	42	45	
J	0	3	6	9	12	15	21	24	27	30	33	36	1 / 12
K	0	3	6	9	12	15	21	24	27	30	33	36	
L	0	3	6	9	12	15	21	24	27	30	33	36	
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Application 3: Asymmetric Players

Game 5 (Payoffs from Game 1 plus 3 ECU in all cells => GGS: entry)

Naive GGS predicts entry of A-C, Level k: A-F

V(i, n)	number of adopters n												observed entries
Player i	1	2	3	4	5	6	7	8	9	10	11	12	
A	33	36	39	39	39	39	39	39	39	39	39	39	12 / 12
B	33	36	39	39	39	39	39	39	39	39	39	39	
C	33	36	39	39	39	39	39	39	39	39	39	39	
D	3	3	3	33	36	39	39	39	39	39	39	39	9 / 12
E	3	3	3	33	36	39	39	39	39	39	39	39	
F	3	3	3	33	36	39	39	39	39	39	39	39	
G	3	3	3	3	3	3	33	36	39	39	39	39	2 / 12
H	3	3	3	3	3	3	33	36	39	39	39	39	
I	3	3	3	3	3	3	33	36	39	39	39	39	
J	3	3	3	3	3	3	3	3	3	33	36	39	3 / 12
K	3	3	3	3	3	3	3	3	3	33	36	39	
L	3	3	3	3	3	3	3	3	3	33	36	39	
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Application 3: Asymmetric Players

Game 6 (Payoffs from Game 2 plus 3 ECU in all cells => GGS: entry)

Naive GGS predicts entry of A-F, Level k: A-F

V(i, n)	number of adopters n												observed
Player i	1	2	3	4	5	6	7	8	9	10	11	12	entries
A	33	36	39	42	45	48	51	54	57	60	63	66	12 / 12
B	33	36	39	42	45	48	51	54	57	60	63	66	
C	33	36	39	42	45	48	51	54	57	60	63	66	
D	3	3	3	33	36	39	42	45	48	51	54	57	7 / 12
E	3	3	3	33	36	39	42	45	48	51	54	57	
F	3	3	3	33	36	39	42	45	48	51	54	57	
G	3	3	3	3	3	3	33	36	39	42	45	48	1 / 12
H	3	3	3	3	3	3	33	36	39	42	45	48	
I	3	3	3	3	3	3	33	36	39	42	45	48	
J	3	3	3	3	3	3	3	3	3	33	36	39	2 / 12
K	3	3	3	3	3	3	3	3	3	33	36	39	
L	3	3	3	3	3	3	3	3	3	33	36	39	
													22 / 48

Application 3: Asymmetric Players

Game 7 (Payoffs from Game 3 plus 3 ECU in all cells => GGS: entry)

Naive GGS predicts entry of A-F, Level k: A-F

V(i, n)	number of adopters n												observed
Player i	1	2	3	4	5	6	7	8	9	10	11	12	entries
A	33	36	39	39	39	39	39	39	39	39	39	39	12 / 12
B	33	36	39	39	39	39	39	39	39	39	39	39	
C	33	36	39	39	39	39	39	39	39	39	39	39	
D	24	27	30	33	36	39	39	39	39	39	39	39	10 / 12
E	24	27	30	33	36	39	39	39	39	39	39	39	
F	24	27	30	33	36	39	39	39	39	39	39	39	
G	12	15	18	24	27	30	33	36	39	39	39	39	5 / 12
H	12	15	18	24	27	30	33	36	39	39	39	39	
I	12	15	18	24	27	30	33	36	39	39	39	39	
J	3	6	9	12	15	18	24	27	30	33	36	39	2 / 12
K	3	6	9	12	15	18	24	27	30	33	36	39	
L	3	6	9	12	15	18	24	27	30	33	36	39	
													29 / 48

Application 3: Asymmetric Players

Game 8 (Payoffs from Game 4 plus 3 ECU in all cells => GGS: entry)

Naive GGS predicts entry of A-F, Level k: A-F

V(i, n)	number of adopters n												observed
Player i	1	2	3	4	5	6	7	8	9	10	11	12	entries
A	33	36	39	42	45	48	51	54	57	60	63	66	11 / 12
B	33	36	39	42	45	48	51	54	57	60	63	66	
C	33	36	39	42	45	48	51	54	57	60	63	66	
D	24	27	30	33	36	39	42	45	48	51	54	57	11 / 12
E	24	27	30	33	36	39	42	45	48	51	54	57	
F	24	27	30	33	36	39	42	45	48	51	54	57	
G	12	15	18	24	27	30	33	36	39	42	45	48	6 / 12
H	12	15	18	24	27	30	33	36	39	42	45	48	
I	12	15	18	24	27	30	33	36	39	42	45	48	
J	3	6	9	12	15	18	24	27	30	33	36	39	1 / 12
K	3	6	9	12	15	18	24	27	30	33	36	39	
L	3	6	9	12	15	18	24	27	30	33	36	39	
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Application 3: Symmetric Players

Game 15 (Payoffs are the same for all players, GGS: entry)

Naive GGS predicts no entry, Level k: no entry

V(i, n)	number of adopters n												observed
Player i	1	2	3	4	5	6	7	8	9	10	11	12	entries
A	8	12	16	20	24	28	32	36	40	44	48	52	
B	8	12	16	20	24	28	32	36	40	44	48	52	
C	8	12	16	20	24	28	32	36	40	44	48	52	
D	8	12	16	20	24	28	32	36	40	44	48	52	
E	8	12	16	20	24	28	32	36	40	44	48	52	
F	8	12	16	20	24	28	32	36	40	44	48	52	
G	8	12	16	20	24	28	32	36	40	44	48	52	
H	8	12	16	20	24	28	32	36	40	44	48	52	
I	8	12	16	20	24	28	32	36	40	44	48	52	
J	8	12	16	20	24	28	32	36	40	44	48	52	
K	8	12	16	20	24	28	32	36	40	44	48	52	
L	8	12	16	20	24	28	32	36	40	44	48	52	
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Application 3: Asymmetric Players

Experiment:

Low types tend to enter, high types tend to stay out.

Comparative static: higher payoffs in cells off the diagonal lead to more entries.

GGS-prediction of same behavior across types does not hold.

○

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○ Behavior may be explained by global-game equilibrium with positive variance (cf. HNO 2009)

or by „naive“ GGS: best response to uniform distribution on the number of others entering (entry if average number in a row >34).

or by levels of reasoning. Level 0: e.b. enters with prob. 50%,
level k best response to level k – 1.
In our games, level k = level 1 for k>1.

Conclusion

- Supermodular games with many actions or asymmetric players can be decomposed into smaller games, for which the global-game selection can be easily derived.
- If solutions for all small games point in direction of one strategy combination, this is the GGSs of the large game.
- Noise independence is inherited from smaller games.
- This allows applying the concept of global games to more complex games.
- GGS cannot explain differences in behavior across different types.