SOME PROPERTIES OF THE AFFINE TOTAL VARIATION USED IN IMAGE SEGMENTATION.

C. Ballester & V. Caselles *
Dept. Matemàtiques. Univ. Illes Balears. 07071 Palma de Mallorca (Spain)

Abstract: *We study the Affine Total Variation, a magnitude measuring the affine complexity of finite unions of continua, in particular, Jordan curves, appearing in an affine invariant analogue of Mumford-Shah energy functional used to segment images. We prove a lower semicontinuity result for the ATV functional.*

Devoted to the memory of Julio Bouillet.

1. INTRODUCTION.

Even if the images we perceive are analyzed and understood without evident effort, the understanding of them involves very complex mechanisms which, by now, we cannot reproduce in a computer. The complexity of image analysis motivated its division in a series of simpler and independent tasks. Among them, edge detection and image segmentation seem to be fundamental. Certainly, we need to identify the objects in a scene and therefore, to find their contours or boundaries. Then, segmenting an image amounts to subdivide the image domain into regions corresponding to the projection of visible surfaces of objects in a real scene. More precisely, on one side, one wishes to smooth the nearly homogeneous regions of the picture with two scopes: noise elimination and image interpretation, and, on the other side, one wants to keep the accurate location of these regions and restore some regularity for their boundaries. A general treatment of this subject can be seen, for instance, in [`MoSoli94`] and [`Rosenfeld`].

Images are the projection of physical objects in the three-dimensional world onto a two-dimensional -planar- surface, be it the retina or an array of sensors in a video camera. Since, in most situations, one cannot control the exact location of the objects to be recognized, we are concerned with finding properties of an image which are invariant to transformations of the image caused by moving an object so as to change its perceived position and orientation. The idea of invariance arises from our ability to recognize objects irrespective of such movement. A good approximation to image formation in a real camera is given by the *perspective camera*

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model in which points are projected from the 3D world onto an image plane so that all rays joining the object and corresponding image points pass through a simple point, called the point of projection. Since, in the perspective camera model, an euclidean motion of a solid object in the 3D world induces a planar projective transformation in the 2D image space, one needs methods or features which are invariant to projective planar transformations. Under the weak perspective assumption, i.e., when the object's depth is small compared with its distance from the camera (which corresponds to the focal distance $f \to \infty$), the planar projective transformations can be approximated by affine linear transformations Hence, we shall look for segmentation methods invariant under affine transformations, as a simplified form of invariance under planar projective transformations. We would like to mention that a lot of interest has been recently given to affine invariant methods in image processing (see [BaCaGon] and its references).

Coming back to our purpose, the recent literature on segmentation problems shows a strong convergence of the methods to variational methods [Mumford], [MoSoli94] (see also [Geman], [Haralick85] for precedents). From these references, it is now well known that a good segmentation can be obtained by minimizing an energy functional. The simplest such energy functional was proposed by Mumford-Shah ([Mumford]). They proposed to segment the image $g : \Omega \to I\!R$ by minimizing

$$E(u, B) = \int_{\Omega \setminus B} \| \nabla u\|^2 + \int_{\Omega} |u - g|^2 + \lambda\, H^1(B), \tag{1.1}$$

where Ω is an open set in $I\!R^2$, generally a rectangle, u is a piecewise smooth function defined on Ω, B is the set of boundaries in Ω –with length $H^1(B)$– where u is discontinuous and $\lambda > 0$. They conjectured in [Mumford] that this functional has a minimum (u, B), with B being a finite set of smooth C^1 curves. The full conjecture has not been proved yet but a lot of significant results have been given ([MoSoli94]). Mumford and Shah also proposed a simplified version, where u is imposed to be a piecewise constant function in $\Omega \setminus B$. In this case, (1.1) writes

$$E(u, B) = \int_{\Omega} |u - g|^2 + \lambda\, H^1(B). \tag{1.2}$$

In [KoeMoSoli], Koepfler-Morel-Solimini proved, mathematically and practically, that the "Region Growing" is an efficient method to minimize this functional (see also [MoSoli94]).

Although the Mumford-Shah functional (1.1) is euclidean invariant, it is not affine invariant. Indeed, the first term and the euclidean length are not invariant by affine transforms. In [BaCaGon], we replaced the euclidean arclength –as a measure of euclidean complexity– by a different expression measuring the affine complexity of the set of boundaries of the segmentation. When thinking in these terms, the first thing coming to mind is the affine length of a curve but this quantity, if thought of as an additive quantity, must be zero for a polygonal curve and does not seem to be the right one if one tries to approximate a smooth curve by a piecewise affine one ([BaCaGon]). The smoothness term in the Mumford-Shah functional (1.1) can also be replaced by an affine invariant one (see [BaCaGon]). In fact, we proposed in [BaCaGon] the following *affine invariant version* of the simplified Mumford-Shah functional (1.2)

$$E_{af}(u,B) = \int_\Omega |u-g|^2 \; + \; \lambda\, ATV(B). \tag{1.3}$$

where u is a piecewise smooth function, B is a family of curves in Ω belonging to a suitable segmentation class and $ATV(B)$ denotes the Affine Total Variation of the segmentation B (see Section 2). Let us briefly explain what each term represents. The first term is the same term that appears in the functional (1.2) expressing the fidelity of the segmentation to the image. Finally, the second term measures the affine complexity of the set of boundaries of the obtained regions. Let us comment that this term is global in nature (i.e., they make all parts of the image interact, no matter their respective distance).

Typically, when minimizing such kind of functionals, we are trying to approximate g by a piecewise smooth function u and, at the same time, to reduce the complexity of the discontinuities of u (the boundaries of the regions in the image). As we analized in [BaCaGon] in the case of (1.3), the discontinuities permitted by the model will be either a finite union of rectifiable curves or a degenerate segmentation composed of a finite or infinite set of parallel lines −this degenerate case can happen (e.g.) if one uses (1.3) to approximate an image which is a linear transition from white to gray.

In [BaCaGon], we studied the affine invariant energy functional (1.3) from a mathematical point of view, stating the existence of minimizers and giving a simple numerical algorithm to minimize it based on the work of [KoeMoSoli] and using also a simple numerical scheme in order to discretize the Affine Total Variation quantity. Our purpose here will be to give a more detailed mathematical analysis of the term $ATV(B)$ introduced in [BaCaGon] to measure the affine complexity of a family of curves. In particular, we extend the ATV magnitude to rectifiable continua (or finite unions of them) and we prove a lower semicontinuity result for the ATV (see Theorem 3.1 below). Even if this has no implications in the context of our assumptions of [BaCaGon] where the admissible segmentations consisted of a finite union of rectifiable Jordan curves with disjoint interiors, it completes the mathematical analysis of the ATV magnitude and some geometrical lemmas used to prove the main result could be interesting by themselves.

Let us explain the plan of the paper. We start in Section 2 by recalling the model and the main results of [BaCaGon]. Then, in Section 3, we shall extend the ATV functional to the natural class of (H^1-rectifiable) continua (and finite unions of them) and prove a lower semicontinuity result for the ATV in this setting.

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2. THE MODEL AND EXISTENCE OF MINIMIZERS.

In this section, we recall the definition of affine total variation of a set of curves and the class of admissible segmentations we used in [BaCaGon] to minimize the proposed functional (1.3). Finally we state without proof the existence of minimizers.

For the sake of definiteness, let Ω be an open rectangle in $I\!R^2$. Let $\lambda > 0$. Let g be the given image, i.e., $g : \Omega \to I\!R_+$ is a bounded measurable function.

We need several definitions to introduce our model. Recall that a *Jordan curve* is a continuous curve $c : [a,b] \to I\!R^2$ such that for all $t, t' \in]a,b[$, $c(t) \neq c(t')$ if $t \neq t'$ $(a < b)$. If $c(a) = c(b)$ the Jordan curve is said to be *closed*. The points $c(a)$ and $c(b)$ will be called *tips* of the curve, all other points in the range of c are *interior points*. Let $\Im$ be the following family of sets

$$\Im = \Big\{ B \subseteq \bar{\Omega} : \quad B \text{ is a finite union of rectifiable Jordan curves whose interiors are disjoint and contained in } \Omega \Big\}.$$

Definition 2.1 *Let $u \in L^2(\Omega)$. We say that u is cylindrical in the direction $v \in I\!R^2, v \neq 0$, if $\nabla u \cdot v = 0$ in the sense of distributions. We say that u is cylindrical if u is cylindrical in some direction $v \in I\!R^2, v \neq 0$.*

A simple argument shows that u is cylindrical in the direction $v \neq 0$ if and only if, after a possible modification of u in a set of null measure, $u(x + \lambda v) = u(x)$ for almost every x and all $\lambda \in [0,1]$, i.e., u is constant on lines parallel to the direction v. Since $u \in L^2(\Omega)$, almost all points $x \in \Omega$ are Lebesgue points of u. To choose a particular representative of u we use the following rule: if for $x \in \Omega$ there exists some $\lambda \in I\!R$ such that

$$\lim_{r \to 0} \frac{1}{\pi r^2} \int_{D(x,r)} |u(y) - \lambda| dy = 0,$$

where $D(x,r) = \{y \in \Omega : ||y - x|| \leq r\}$, then we define $u(x) = \lambda$. Hence when, for a cylindrical function, we speak of the discontinuity set of u we mean the discontinuity set of its chosen representative.

Let

$$\zeta_0 := \Big\{ u : \quad \text{there exists } B \in \Im \text{ such that } u : \Omega \to I\!R_+ \text{ is constant on each connected component of } \Omega \setminus B \text{ and } u \text{ is discontinuous on } B \Big\}$$

$$\zeta_1 := \Big\{ u : \quad u : \Omega \to I\!R_+ \text{ is a cylindrical function } \Big\}.$$

Let

$$\zeta = \zeta_0 \cup \zeta_1.$$

It will be common to call members u of ζ *segmentations.* Sometimes we will also refer to function u as the *segmented image* and its discontinuity set B as the *segmentation boundaries* or, simply, *segmentation.* Let us observe that segmentations in ζ_0 are Mumford-Shah type segmentations while segmentations in ζ_1 are affine degenerate segmentations. This would correspond to a underlying transformation of the image by a linear map A with one of the eigenvalues near to zero.

To introduce the $ATV(\cdot)$, let us define:

Definition 2.2 *Let $\Gamma, \tilde{\Gamma}$ be two rectifiable Jordan curves. We define the interaction of Γ and $\tilde{\Gamma}$ by*

$$Inter(\Gamma, \tilde{\Gamma}) = \int_{\Gamma} \int_{\tilde{\Gamma}} |\tau(x) \wedge \tilde{\tau}(y)| \, d\sigma(x) \, d\tilde{\sigma}(y), \tag{2.1}$$

where $\sigma, \tilde{\sigma}$ denote, respectively, the arclength parameters on each curve $\Gamma, \tilde{\Gamma}$ and $\tau(x), \tilde{\tau}(y)$ denote the tangent vectors at $x \in \Gamma$ and $y \in \tilde{\Gamma}$, respectively.

For convenience in notation, given $u \in \zeta_0$, let us consider B as the set of discontinuity of u and write $(u, B) \in \zeta_0$ instead of $u \in \zeta_0$. If u is in ζ_1, the discontinuity set of u may be very wild. On the other hand, it will not play any role in what follows. But, for a uniform notation below, it will be convenient to write also B as the discontinuity set of u and write $(u, B) \in \zeta_1$ instead of $u \in \zeta_1$. We also refer to pairs $(u, B) \in \zeta$ as segmentations.

We now define the ATV functional. Let $(u, B) \in \zeta$. If $(u, B) \in \zeta_0$, then $B = \bigcup_{i=1}^{N} \Gamma_i$ where Γ_i are rectifiable Jordan curves whose interiors are disjoint. We set

$$ATV(B) = \sum_{i,j=1}^{N} Inter(\Gamma_i, \Gamma_j).$$

If $(u, B) \in \zeta_1$, then we set $ATV(B) = 0$. In any case, we define

$$E_{af}(u, B) = \int_{\Omega} |u - g|^2 \; + \; \lambda \, ATV(B) \tag{1.3}$$

and we want to minimize it on the class of segmentations ζ.

With these definitions, *Functional (1.3) is affine invariant.* Moreover, as proved in [BaCaGon] the ATV functional is the only positive functional, up to a scaling factor, associating to each pair of Jordan curves a quantity which is geometric, affine invariant, biadditive and continuous (in the $W^{1,1}$ topology of the space of parametric curves). With these preliminaries we have:

Theorem 2.1 *E_{af} attains its infimum at some $(u, B) \in \zeta$.*

The proof of Theorem 2.1, which, as usual, is based on a lower semicontinuity result of the energy functional, can be seen in [BaCaGon].

3. LOWER SEMICONTINUITY OF THE ATV FUNCTIONAL IN A MORE GENERAL FRAMEWORK.

In this section we prove that the Affine Total Variation is a lower semicontinuous functional on a wider class of sets, more specifically, the class of sets made of a finite union of (H^1-rectifiable) continua. Let us recall some definitions and terminology. Let Ω be an open connected set in $I\!R^2$ whose boundary is a smooth Jordan curve. We start with some basic notions of geometric measure theory which will be needed to introduce the current setting. Recall that a continuum is a compact connected set with finite H^1-measure. Given a continuum E, by $H^1(E)$ we denote the 1-dimensional Hausdorff measure of E. It can be proved (see [Falconer], [MoSoli94]) that a continuum is the union of a negligible set F_0 (with $H^1(F_0) = 0$) and of a finite or countable union of curves which form an arcwise connected set –i.e., any two points of E may be connected by an arc contained in the continuum– and $H^1(E)$ is the sum of the lengths of this system of curves. A detailed account of it is given in [Falconer] or [MoSoli94].

Definition 3.1 *Let $E, \tilde{E}$ be two continua. Then each one consists of a countable union of rectifiable curves, together with a set of H^1-measure zero. Let $E = F_0 \bigcup \Big(\bigcup_{i=1}^{\infty} \Gamma_i \Big)$, $\tilde{E} = \tilde{F}_0 \bigcup \Big(\bigcup_{i=1}^{\infty} \tilde{\Gamma}_i \Big)$ be such decompositions, where $\Gamma_i, \tilde{\Gamma}_i$ are rectifiable Jordan curves with $\Gamma_i \cap \Gamma_j = \tilde{\Gamma}_i \cap \tilde{\Gamma}_j = \emptyset$ for $i \neq j$ and $H^1(F_0) = H^1(\tilde{F}_0) = 0$. Then, we define the* **Interaction of** *$E, \tilde{E}$ by*

$$Inter(E, \tilde{E}) = \sum_{i,j=1}^{\infty} Inter(\Gamma_i, \tilde{\Gamma}_j),$$

where $Inter(\Gamma_i, \tilde{\Gamma}_j)$ is given as in Definition 2.2.

Let $\Im$ be the following family of sets

$$\Im = \Big\{ B \subset R^2 : \quad B = \bigcup_{finite} B^k, \ B^k \text{continuum}, \ B^k \cap B^j = \emptyset, \ k \neq j \Big\}.$$

Given $B \in \Im$, we define the **Affine Total Variation of** B by

$$ATV(B) = \sum_{k,j} Inter(B^k, B^j).$$

Recalling that, for $B \in \Im$, the tangent vector τ can be defined as a vector measure on B, $d\tau(x)$, with a vector density with respect to the Hausdorff measure H^1 with values in S^1, a more compact and intrinsic definition of ATV

$$ATV(B) = \int_B \int_B |d\tau(x) \wedge d\tau(y)| = \int_B \int_B |\tau(x) \wedge \tau(y)| d\sigma(x) d\sigma(y)$$

makes sense.

Given $v \in I\!R^2$, it is clear what the notation

$$\int_B |v \wedge \tau(y)| d\sigma(y)$$

means if $B \subseteq \bigcup_{k=1}^{N} B^k$ where B^k are continua with $B^k \cap B^j = \emptyset$, $k \neq j$. Indeed, for each $k = 1, ..., N$ let, as above, $B^k = F_0^k \bigcup \left(\bigcup_{i=1}^{\infty} \Gamma_i^k \right)$, where Γ_i^k are disjoint rectifiable Jordan curves with $\Gamma_i^k \cap \Gamma_j^k = \emptyset$ for $i \neq j$ and $H^1(F_0^k) = 0$. We define

$$\int_B |v \wedge \tau(y)| d\sigma(y) = \sum_{k=1}^{N} \sum_{i=1}^{\infty} \int_{B \cap \Gamma_i^k} |v \wedge \tau(y)| d\sigma(y).$$

Our purpose is to state the lower semicontinuity of the ATV functional with respect to the Hausdorff distance.

Definition. *Given a sequence $\{B_n\} \subset \Im$ and $B \in \Im$, we shall say that the sequence B_n* **converges to** *B if B_n converges to B in the Hausdorff metric.*

Theorem 3.1 *Let B_n be a sequence in $\Im$ such that $ATV(B_n) \leq M$ for all n and $\sup_n c(B_n) < +\infty$, where $c(B_n)$ denotes the cardinal of continua contained in B_n. Then, there exists a subsequence, still called B_n, and $B \in \Im$, such that B_n converges to B and*

$$ATV(B) \leq \liminf_{n \to \infty} ATV(B_n).$$

To prove Theorem 3.1 we start with two lemmas which have the following geometrical interpretation: either the sequence B_n tends to a segmentation containing two linearly independent directions or the segmentations B_n tend to oscillate in a single direction giving in the limit a degenerate segmentation.

Lemma 3.2 *Let $\{B_n\}$ be a sequence in $\Im$. Then, either*

(a) $\exists \eta > 0$ such that $Inter(\delta, B_n) \geq \eta H^1(\delta)$ for all n and all Jordan curves δ whose range is contained in B_n, or

(b) there exists a subsequence of $\{B_n\}$, still called $\{B_n\}$, and vectors $v_n \in I\!R^2$, $\|v_n\| = 1$, such that

$$\int_{B_n} |v_n \wedge \tau(y)| d\sigma(y) \to 0 \qquad as \quad n \to \infty.$$

Proof. If

(3.1) $\exists \eta > 0$ such that $\forall n \in I\!N$, $\forall v \in I\!R^2$ with $\|v\| = 1$,

$$\int_{B_n} |v \wedge \tau(y)| d\sigma(y) \geq \eta,$$

then (a) immediately follows. In case (3.1) is not true, then

$\forall m \in I\!N$, there exist $n_m \in I\!N$ and $v_m \in I\!R^2$ with $\|v_m\| = 1$ satisfying

$$\int_{B_{n_m}} |v_m \wedge \tau(y)| d\sigma(y) \leq \frac{1}{m},$$

which gives the statement (b) above. ∎

Lemma 3.3 *Let $\{B_n\}$ be a sequence in $\Im$ such that $ATV(B_n) \leq M$ for all n. Then there exists a subsequence, still called $\{B_n\}$, such that*

either (i) $\sup_n H^1(B_n) \leq C$,

or (ii) ("degeneration") there exists a vector $v \in R^2$ such that $\forall \rho > 0$

$$H^1\big(\{x \in B_n : |sin(\tau(x), v)| < \rho\}\big) \to +\infty$$

and

$$H^1\big(\{x \in B_n : |sin(\tau(x), v)| \geq \rho\}\big) \to 0.$$

Proof. From the proof of previous Lemma 3.2, we have

either $\exists \eta > 0$ such that $\forall n \in I\!N$, $\forall v \in I\!R^2$ with $\|v\| = 1$,

$$\int_{B_n} |v \wedge \tau(y)| d\sigma(y) \geq \eta \tag{3.2}$$

or, $\forall m \in I\!N$, there exist $n_m \in I\!N$ and $v_m \in I\!R^2$ with $\|v_m\| = 1$ satisfying

$$\int_{B_{n_m}} |v_m \wedge \tau(y)| d\sigma(y) \leq \frac{1}{m}. \tag{3.3}$$

In the case of (3.2) , we obtain

$$ATV(B_n) = \int_{B_n} \int_{B_n} |\tau(x) \wedge \tau(y)| d\sigma(x) d\sigma(y) \geq \eta H^1(B_n). \tag{3.4}$$

Since $ATV(B_n) \leq M$ for all n, (3.4) yields part (i) of the lemma.

In case (3.2) is not true, we have (3.3). Let us denote the subsequence B_{n_m} again by B_n. Then, for any $\rho > 0$,

$$\begin{aligned} \int_{B_n} |v_n \wedge \tau(x)| d\sigma(x) &= \int_{B_n} |sin(v_n, \tau(x))| d\sigma(x) \\ &\geq \int_{\{x \in B_n : |sin(\tau(x), v_n)| \geq \rho\}} |sin(v_n, \tau(x))| d\sigma(x) \\ &\geq \rho H^1\big(\{x \in B_n : |sin(\tau(x), v_n)| \geq \rho\}\big), \end{aligned}$$

which implies that $H^1\big(\{x \in B_n : |sin(\tau(x), v_n)| \geq \rho\}\big) \to 0$ as $n \to \infty$.

On the other hand, since we can assume, without loss of generality, that $H^1(B_n) \to +\infty$ as $n \to \infty$ and

$$H^1(B_n) = H^1(\{x \in B_n : |sin(\tau(x), v_n)| < \rho\}) + H^1(\{x \in B_n : |sin(\tau(x), v_n)| \geq \rho\})$$

we obtain that $H^1\big(\{x \in B_n : |sin(\tau(x), v_n)| < \rho\}\big) \to +\infty$ as $n \to \infty$, $\forall \rho > 0$.

Now, there exists a subsequence of $\{v_n\}$, still called $\{v_n\}$, and a vector v, $\|v\| = 1$, such that $v_n \to v$. Take $\epsilon > 0$. Let n_0 be such that $|sin(v_n, v)| < \epsilon$, for all $n \geq n_0$. By elementary trigonometry,

$$|sin(\tau(x), v)| \leq |sin(\tau(x), v_n)| + |sin(v_n, v)| < \rho + \epsilon,$$

for $x \in \{y \in B_n : |sin(\tau(y), v_n)| < \rho\}$, $\rho > 0$. From that, the set $\{x \in B_n : |sin(\tau(x), v_n)| < \rho\}$ is included in $\{x \in B_n : |sin(\tau(x), v)| < \rho + \epsilon\}$ if $n \geq n_0$. Thus $H^1\big(\{x \in B_n : |sin(\tau(x), v)| < \rho + \epsilon\}\big) \to +\infty$, as $n \to \infty, \forall \epsilon > 0$, which gives the first statement in (ii).

To prove the second statement it is sufficient to follow the same argument as above, observing that

$$\begin{aligned} |sin(\tau(x), v_n)| &\geq |sin((\tau(x), v) - (v_n, v))| \\ &\geq |sin(\tau(x), v)||cos(v_n, v)| - |cos(\tau(x), v)||sin(v_n, v)|. \end{aligned}$$ ∎

Lemma 3.4 *Suppose that ("degeneration") of Lemma 3.3 holds. Let $g_n : [0, L_n] \to R^2$ be a curve parametrized by its arclength whose image $Img_n \subseteq B_n$. Extend g_n to $f_n : [0, +\infty[\to R^2$ by $f_n(s) = g_n(L_n)$ for $s \geq L_n$. Then, there exists a subsequence of $\{f_n\}$, called again $\{f_n\}$, and a function $f : [0, +\infty[\to R^2$ parametrizing a line segment in the direction v such that*

(3.5) $\quad f_n \to f \quad$ *in* $C_{loc}([0, \infty[)$,

(3.6) $\quad f_n' \to f' \quad$ *in the weak*topology* $\sigma(L^\infty([0, \infty[), L^1([0, \infty[))$.

Remark 3.1. It follows from the statement of Lemma 3.4 that if $\sup L_n < +\infty$ then Imf_n converges to Imf in the Hausdorff topology.

Proof. Since $Imf_n \subseteq B_n$, the range of f_n is bounded. Moreover, $\sup\limits_n \|f_n'\|_\infty < +\infty$. Then, there exists a subsequence of $\{f_n\}$, called again $\{f_n\}$, and a function $f : [0, +\infty[\to R^2$ such that (3.5) and (3.6) hold. Now we write: $f_n' = (f_n' \cdot v)\, v + (f_n' \cdot v^\perp)\, v^\perp$, where v is the vector coming from ("degeneration") in Lemma 3.3 , with $\|v\| = 1$. We are going to prove that

(3.7) $$f_n' \cdot v^\perp \to 0 \text{ in } \sigma\left(L^\infty([0, \infty[), L^1([0, \infty[)\right).$$

To this aim, we estimate $\int_0^L |f_n'(s) \cdot v^\perp|\, ds$ for any $L > 0$. Let $L > 0$.

If $L_n \to +\infty$, taking n large enough we may assume that $L_n > L$. Then

$$\begin{aligned} \int_0^L |f_n'(s) \cdot v^\perp|\, ds &= \int_0^L |\sin(f_n'(s), v)|\, ds \\ &= \int_{\{s \in [0,L] : |sin(f_n'(s), v)| < \rho\}} |sin(f_n'(s), v)|\, ds \\ &\quad + \int_{\{s \in [0,L] : |sin(f_n'(s), v)| \geq \rho\}} |sin(f_n'(s), v)|\, ds \\ &\leq \rho L + \int_{\{s \in [0,L] : |sin(f_n'(s), v)| \geq \rho\}} ds. \end{aligned}$$

Since ("degeneration") of Lemma 3.3 holds,

$$0 \leq \limsup_{n\to\infty} \int_0^L |f_n'(s)\cdot v^\perp|\, ds \;\leq\; \rho\, L. \tag{3.8}$$

Since this is true for any $\rho > 0$, it follows that

$$f_n'(s)\cdot v^\perp \to 0 \quad in\ L^1[0,L]. \tag{3.9}$$

If $\sup L_n < +\infty$, take $L > \sup L_n$. Since $f_n'(s) = 0$ for any $s > L_n$,

$$\int_0^L |f_n'(s)\cdot v^\perp|\, ds = \int_0^{L_n} |f_n'(s)\cdot v^\perp|\, ds = \int_0^{L_n} |\sin(f_n'(s), v)|\, ds.$$

As above we prove that (3.8) and (3.9) follow. Now let $g \in L^1[0,+\infty[$. Then for any $L, N > 0$,

$$\begin{aligned}
&\left|\int_0^{+\infty} f_n'(s)\cdot v^\perp g(s)ds\right| \leq \int_0^{+\infty} |f_n'(s)\cdot v^\perp||g(s)|ds \\
&= \int_0^L |f_n'(s)\cdot v^\perp| \inf(|g(s)|,N)ds + \int_0^L |f_n'(s)\cdot v^\perp|(|g(s)|-N)^+ds \\
&+ \int_L^{+\infty} |f_n'(s)\cdot v^\perp||g(s)|ds \leq N\int_0^L |f_n'(s)\cdot v^\perp|ds \\
&+ \int_0^L (|g(s)|-N)^+ds + \int_L^{+\infty} |g(s)|ds.
\end{aligned}$$

Letting $n\to\infty$

$$\limsup_{n\to\infty}\left|\int_0^{+\infty} f_n'(s)\cdot v^\perp\cdot g(s)ds\right| \leq \int_0^L (|g(s)|-N)^+ds + \int_L^{+\infty} |g(s)|ds.$$

Letting $N\to\infty$ and $L\to\infty$ in this order in the above expression we get

$$\limsup_{n\to\infty}\int_0^{+\infty} f_n'(s)\cdot v^\perp g(s)ds = 0.$$

This proves (3.7). It follows that

$$f_n' = (f_n'\cdot v)v + (f_n'\cdot v^\perp)v^\perp \rightharpoonup (f'\cdot v)v \text{ in } \sigma\left(L^\infty([0,\infty[), L^1([0,\infty[)\right).$$

Since, on the other hand, $f_n' \rightharpoonup f'$ in that topology, we get $f' = (f'\cdot v)\,v$. Hence, $f'(s) = \lambda(s)\,v$, where $\lambda \in L^\infty[0,+\infty[$, i.e.

$$f(t) = f(0) + \Big(\int_0^t \lambda(s)ds\Big)\,v.$$

f parametrizes a segment in the direction v. ∎

The following simple technical fact will be required.

Lemma 3.5 *Let $p, q \in \Omega$. Let $[p, q]$ be the segment joining both points, i.e. $[p, q] = \{tp + (1-t)q : t \in [0, 1]\}$, and let δ be any Jordan curve joining p and q. Then, for any $B \in \Im$*

$$Inter(B, \delta) \geq Inter(B, [p, q]).$$

Proof. Without loss of generality we may assume that B is a Jordan curve in Ω. Since for any $x \in [p, q]$, $\tau(x) = \frac{p-q}{||p-q||}$

$$\begin{aligned} Inter(B, [p, q]) &= \int_{[p,q]} \int_B |\tau(x) \wedge \tau(y)| d\sigma(y) d\sigma(x) = \int_B |(p-q) \wedge \tau(y)| d\sigma(y) \\ &= \int_B \left| \int_\delta \tau(x) d\sigma(x) \wedge \tau(y) \right| d\sigma(y) \leq \int_B \int_\delta |\tau(x) \wedge \tau(y)| d\sigma(x) d\sigma(y) \\ &= Inter(B, \delta). \end{aligned}$$ ∎

Lemma 3.6 *Suppose that ("degeneration") of Lemma 3.3 holds. Moreover, suppose that $\sup_n c(B_n) < +\infty$, where $c(B_n)$ denotes the cardinal of continua contained in B_n. Then, there exists a subsequence of B_n, called again B_n, such that B_n converges to B where $B \in \Im$ consists of a finite number of line segments parallel to v (which may possibly be reduced to a point).*

Proof. Since $B_n \in \Im$ and $\sup_n c(B_n) < +\infty$, there exists a subsequence B_n such that $c(B_n) = k$ for all n and we may write $B_n = K_{n1} \cup ... \cup K_{nk}$, where K_{ni} are continua with $H^1(K_{ni}) < +\infty$ and $K_{ni} \cap K_{nj} = \emptyset$ for $i \neq j$. Our strategy will be as follows. We take $i = 1$ and construct a subsequence of K_{n1} converging to a line segment parallel to v (possibly reduced to a point). Having constructed a subsequence $\{n_r\}$ of $I\!N$ such that $K_{n_r i}$ converges to a line segment parallel to v for any $i = 1, 2, ..., j-1\,(j \leq k)$ we take $i = j$ and construct a subsequence $\{n_{r_l}\}$ of $\{n_r\}$ such that $K_{n_{r_l} j}$ also converges to a line segment parallel to v. Our lemma follows from this construction. Our proof reduces to a single step. Suppose that K_{ni}, $i < j$, converges to a line segment parallel to v. Consider $i = j$. Using the Blaschke selection theorem, we find a subsequence of K_{nj}, call it again K_{nj}, such that $K_{nj} \to K_j$ in the Hausdorff distance where K_j is a continuum. If K_j is not reduced to a point, we find points $p, q \in K_j$, $p_n, q_n \in K_{nj}$ such that $p_n \to p$, $q_n \to q$ and $||p_n - q_n|| \geq \alpha > 0$ for all n, for some $\alpha > 0$. Since K_{nj} is a continuum, there exists an arc $[p_n, q_n] \subseteq K_{nj}$ joining p_n to q_n. By Lemma 3.4, $[p_n, q_n]$ can be suitably parametrized to converge to a line segment S_{jv} in the weak* topology $\sigma(L^\infty[0, +\infty[, L^1[0, +\infty[)$. Set $B_{jv} = K_j \cap \{$line passing through a point in S_{jv} in the direction $v\}$. (Observe that $p \in S_{jv}$.) We claim that $K_j = B_{jv}$. Otherwise, there exists a point $\tilde{p} \in K_j$ such that $d(\tilde{p}, B_{jv}) > 0$. As above we may find $p_n, q_n \in K_{nj}$ such that $p_n \to \tilde{p}$, $q_n \to \tilde{q} \in B_{jv}$, $||p_n - q_n|| \geq \alpha > 0$ for all n, for some $\alpha > 0$. Let $u_n = p_n - q_n$. Let $[p_n, q_n]$ be an arc contained in K_{nj} joining

p_n to q_n. If $\sup\limits_n H^1(K_{nj}) < +\infty$, the length of $[p_n, q_n]$ is uniformly bounded. By Lemma 3.4 and Remark 3.1, we know that, after extracting a subsequence, $[p_n, q_n]$ converges in the Hausdorff distance to a line segment L parallel to v. It follows that $\tilde{p}, \tilde{q} \in L$. Hence $\tilde{p} \in B_{jv}$, which yields a contradiction. In this case, $K_j = B_{jv}$. Now we may assume that $H^1(K_{nj}) \to +\infty$ as $n \to \infty$. We also assume that $u_n \to u$ where $\|u\| \geq \alpha > 0$ is not parallel to v. Choosing ρ sufficiently small we may assume that

(3.10)
$$|\sin(\tau(x), u)| \geq \eta > 0 \text{ for some } \eta > 0 \text{ and all } x \in \{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}.$$

Finally, recall that, by Lemma 3.3, we may suppose that $H^1(\{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}) \to +\infty$ as $n \to \infty$. Now, set $K_{nj} = \bigcup\limits_m \Gamma_{nj_m} \bigcup F_{nj_0}$, where Γ_{nj_m} are rectifiable curves and $H^1(F_{nj_0}) = 0$. Let

$$\begin{aligned} A_{nj} &\equiv Inter\big(\{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}, u_n\big) \\ &\equiv \sum_m Inter\big(\{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\} \cap \Gamma_{nj_m}, u_n\big) \\ &= \sum_m \int_{\Gamma_{nj_m} \cap \{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}} |\tau(x) \wedge u_n| \, d\sigma_m(x), \end{aligned}$$

where $\sigma_m(x)$ denotes the arclenth of the curve Γ_{nj_m}. Since, by Lemma 3.5,

$$|\tau(x) \wedge u_n| \leq |\tau(x) \wedge \int_{[p_n, q_n]} \tau(y) \, d\sigma(y)| \leq \int_{[p_n, q_n]} |\tau(x) \wedge \tau(y)| \, d\sigma(y),$$

we have

(3.11)
$$\begin{aligned} &A_{nj} \leq \sum_j \int_{\Gamma_{nj_m} \cap \{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}} \int_{[p_n, q_n]} |\tau(x) \wedge \tau(y)| \, d\sigma(y) \, d\sigma_m(x) \\ &= Inter\big(\{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}, [p_n, q_n]\big) \leq ATV(K_{nj}) \leq ATV(B_n) \leq M. \end{aligned}$$

On the other hand,

$$\begin{aligned} A_{nj} &= \sum_m \int_{\Gamma_{nj_m} \cap \{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}} |\tau(x) \wedge u_n| \, d\sigma_m(x) \\ &= \|u_n\| \sum_m \int_{\Gamma_{nj_m} \cap \{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}} |\sin(\tau(x), u_n)| d\sigma_m(x). \end{aligned}$$

Using (3.10) ,

$$\begin{aligned} A_{nj} \geq &\|u_n\| \, \eta \sum_m \int_{\Gamma_{nj_m} \cap \{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}} d\sigma_m(x) \\ = &\eta \, \|u_n\| \, H^1\big(\{x \in K_{nj} : |\sin(\tau(x), v)| < \rho\}\big). \end{aligned} \tag{3.12}$$

As observed above, the right hand side of (3.12) tends to $+\infty$ as $n \to \infty$, contradicting (3.11). We have proved that $K_j = B_{jv}$. Our lemma is proved. ∎

If we may expect the lower semicontinuity result of Theorem 3.1 to be true, the same result should be true for Jordan curves. Indeed, this is the case and it is stated in the next Lemma which will be needed during the proof of Theorem 3.1.

Lemma 3.7 *([BaCaGon] Lemma 4.7) Let $f_n : [0, L_{n1}] \to \mathbb{R}^2$, $g_n : [0, L_{n2}] \to \mathbb{R}^2$ be the arclength parametrizations of sequences of Jordan curves $A_n = f_n([0, L_{n1}])$, $B_n = g_n([0, L_{n2}])$. Suppose that L_{n1}, L_{n2} are bounded sequences. Suppose that $A_n \to A$, $B_n \to B$ in the Hausdorff distance. Then*

$$Inter(A, B) \leq \liminf_n Inter(A_n, B_n). \tag{3.13}$$

The proof of Theorem 3.1 will be a consequence of the following geometrical result which may be interesting by itself.

Lemma 3.8 *Let K_j be a sequence of continua such that $\sup_j H^1(K_j) < +\infty$ and $K_j \to K$ as $j \to \infty$. Let $C_1, \ldots, C_p$ be a system of Jordan curves such that $C_i \subseteq K$, $C_i \cap C_j = \emptyset$, $i \neq j$, $i, j = 1, \ldots, p$. Then, there exists a sequence $\{j_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ and sequences of curves $\{D_n^i\}_{n=1}^{\infty}$, $i = 1, \ldots, p$, such that*

$$D_n^i \subseteq C_i + B\big(0, \frac{1}{n}\big), \qquad i = 1, ..., p, \tag{3.14}$$

$$C_i \subseteq D_n^i + B\big(0, \frac{1}{n}\big), \qquad i = 1, ..., p, \tag{3.15}$$

$$H^1\big(\bigcup_{i=1}^{p} D_n^i \setminus K_{j_n}\big) \leq \frac{18}{n}\big(H^1(K) + 1\big). \tag{3.16}$$

Hence, (3.14), (3.15), (3.16) imply that, for each $i = 1, ..., p$, we may construct a sequence of curves D_n^i contained in K_{j_n} up to a set of small H^1-measure and such that $D_n^i \to C_i$ as $n \to \infty$.

To prepare the geometrical construction needed for the proof of Lemma 3.8, we recall the following result which was pointed to us by J.M. Morel.

Lemma 3.9 *([MoSoli94], 9.28, 9.31, 9.57). Let K be a regular 1-set (for instance, a continuum). Then, there exists $K' \subseteq K$ with $H^1(K \setminus K') = 0$ such that for all $x \in K'$ there exists a line $D(x)$ such that*

$$\forall \epsilon > 0, \forall r_0 > 0, \exists r < r_0 \textit{ such that}$$

$$H^1\big(P_{D(x)}\big(K \cap B(x, r)\big)\big) \geq (1 - \epsilon)2r \tag{3.17}$$

$$H^1\left((B(x, r) \setminus D(x, r, \epsilon r)) \cap K\right) < \epsilon r \tag{3.18}$$

$$H^1\big(B(x, r) \cap K\big) \leq (2 + \epsilon)r \tag{3.19}$$

where $P_{D(x)}$ *(respectively* $P_{D(x)^\perp}$ *) denotes the projection onto the line* $D(x)$ *(respectively* $D(x)^\perp$*, the orthogonal to* $D(x)$ *passing through* x *) and* $D(x,r,a) = \{y \in B(x,r) \quad \|P_{D(x)^\perp}(y-x)\| \le a\}$.

Proof of Lemma 3.8. Let $d^* = \inf\{d(C_i, C_j) : i, j = 1, 2, \ldots, p,\ i \ne j\}$. Fix $\epsilon \in (0,1)$ and $r_0 < \frac{d^*}{4}$, $r_0 > 0$. Consider the family $V(\epsilon, r_0) = \{\, B(x,r) : 0 < r < r_0, x \in K'$, $B(x,r)$satisfies (3.17), (3.18), (3.19) $\}$. It is clear by Lemma 3.9 that $V(\epsilon, r_0)$ is a Vitali covering of K'. Then, we select a finite or countable disjoint sequence $F = \{B(x_j, r_j)\}_{j=1}^\infty \subset V(\epsilon, r_0)$ such that $H^1(K \setminus \cup_{j=1}^\infty B(x_j, r_j)) = 0$ and $H^1(K) \le \sum_{j=1}^\infty 2r_j + \epsilon$. Observe that, by our choice of r_0, no ball of F intersects two of the curves $C_1, C_2, \ldots, C_p$. Moreover, since K is connected, we have

$$\widetilde{(3.18)} \qquad H^1\left((B(x_j,(1-\epsilon)r_j) \setminus D(x_j,(1-\epsilon)r_j, 2\epsilon r_j)) \cap K\right) = 0, \qquad \forall j \in I\!N.$$

In fact, since K is connected, if there were a point of K in $B(x_j,(1-\epsilon)r_j) \setminus D(x_j,(1-\epsilon)r_j, 2\epsilon r_j)$ there would exist an arc joining it to $K \cap D(x_j,(1-\epsilon)r_j, \epsilon r_j)$. This would imply the existence of an arc of K of length at least ϵr_j crossing either $D(x_j,(1-\epsilon)r_j, 2\epsilon r_j) \setminus D(x_j,(1-\epsilon)r_j, \epsilon r_j)$ or $B(x_j, r_j) \setminus B(x_j,(1-\epsilon)r_j)$. This would contradict (3.18).

Let us also observe that it follows from (3.17)

$$\widetilde{(3.17)} \qquad H^1\left(P_{D(x)}(K \cap B(x_j,(1-\epsilon k)r_j))\right) \ge (1-\epsilon)2r_j - 2k\epsilon r_j$$

for all k such that $k\epsilon < 1$ and $j \in I\!N$. On the other hand, observe that

$$\begin{aligned}\sum_{j=1}^\infty 2r_j &\le \frac{1}{1-\epsilon}\sum_{j=1}^\infty H^1(P_{D(x_j)}(K \cap B(x_j, r_j))) \\ &\le \frac{1}{1-\epsilon}\sum_{j=1}^\infty H^1(K \cap B(x_j, r_j)) \le \frac{H^1(K)}{1-\epsilon} < +\infty.\end{aligned}$$

Choose $\rho > 0$, $\rho < min\big\{\epsilon, \min\{H^1(C_i) : i = 1, ..., p\}\big\}$. Let $N = N(\rho)$ be such that $\sum\limits_{j=N+1}^\infty 2r_j < \rho$. To simplify our notation, let us write $r_{jk} = (1-k\epsilon)r_j$. Let us define the familly of balls:

$$F_{ik} = \big\{B(x_j, r_{jk}) : j \le N,\ B(x_j, r_j) \in F,\ B(x_j, r_{jk}) \cap C_i \ne \emptyset\big\},$$

$i = 1, 2, ..., p$, k such that $k\epsilon < 1$.

For the sake of simplicity let us concentrate our argument on one of the curves C_i, $i = 1, 2, \ldots, p$, say on C_1. Fix a parametrization of C_1. We claim that for $k = 7$, we may renumber the balls of F_{17}

$$(3.20) \qquad F_{17} = \{B(x_j, r_{j7}) : j = 1, \ldots, N_{17}\}$$

so that if $j_1 < j_2$ then C_1 enters $B(x_{j_1}, r_{j_1 7})$ before it enters $B(x_{j_2}, r_{j_2 7})$ and it does not enter again $B(x_{j_1}, r_{j_1 7})$ after $B(x_{j_2}, r_{j_2 7})$.

For that, for each ball $B = B(x_j, r_{j7}) \in F_{17}$, let $p(x_j, r_{j7})$, $q(\underset{\smile}{x_j}, r_{j7})$ be the first and last point of C_1 in B respectively. Observe that, by (3.18), when C_1 enters $B(x_j, r_{j1})$ or $B(x_j, r_{j7})$ it does it through $D(x_j, r_{j1}, 2\epsilon r_j)$ or $D(x_j, r_{j7}, 2\epsilon r_j)$ respectively. Let us observe that

$$\text{(3.21)} \quad \text{from} \quad p(x_j, r_{j7}) \text{ to } q(x_j, r_{j7}),\ C_1 \text{ is entirely contained in } B(x_j, r_{j1}).$$

Else, this would imply a cost in length for C_1, hence for K, in $B(x_j, r_{j1}) \backslash B(x_j, r_{j7})$ of, at least,

$$\text{(3.22)} \qquad 3 \cdot 6\epsilon r_j = 18\epsilon r_j.$$

On the other hand, since, by using (3.17)

$$\begin{aligned} H^1(K \cap B(x_j, r_{j7})) &\geq H^1(P_{D(x_j)}(K \cap B(x_j, r_{j7}))) \geq H^1(P_{D(x_j)}(K \cap B(x_j, r_j))) \\ &\quad - H^1(P_{D(x_j)}(K \cap (B(x_j, r_j) \setminus B(x_j, r_{j7})))) \\ &\geq (1-\epsilon)2r_j - 2 \cdot 7\epsilon r_j = 2r_j - 16\epsilon r_j \end{aligned}$$

and, using (3.19)

$$\text{(3.23)} \quad \begin{aligned} H^1(K \cap (B(x_j, r_j) \setminus B(x_j, r_{j7}))) &\leq (2+\epsilon)r_j - H^1(K \cap B(x_j, r_{j7})) \\ &\leq (2+\epsilon)r_j - 2r_j + 16\epsilon r_j = 17\epsilon r_j. \end{aligned}$$

This contradicts our previous estimate (3.22). Therefore (3.21) follows. In particular, C_1 does not visit another ball in between $p(x_j, r_{j_7})$ and $q(x_j, r_{j_7})$. With these remarks, we may renumber the balls in F_{17} as in (3.20) so that $(1 \leq) j_1 < j_2 (\leq N_{17})$ if and only if C_1 enters $B(x_{j_1}, r_{j_1 7})$ before it enters $B(x_j, r_{j_2 7})$. As we have shown above, if $j_1 < j_2$ we cannot go back to $B(x_{j_1}, r_{j_1 7})$ after going to $B(x_{j_2}, r_{j_2 7})$.

Now, it is clear that $\partial B(x_j, r_{j7}) \cap D(x_j, r_{j7}, 2\epsilon r_j)$ has two connected components. Call $l(x_j, r_{j7})$ the connected component containing $p(x_j, r_{j7})$ and call $R(x_j, r_{j7})$ the other one. Let $cl(x_j, r_{j7}) = \{p \in D(x_j, r_{j7}, 2\epsilon r_{j7}) : p \text{ is connected to } l(x_j, r_{j7}) \text{ by an arc of } C_1 \text{ contained in } D(x_j, r_{j7}, 2\epsilon r_{j7})\}$, $cR(x_j, r_{j7}) = \{p \in D(x_j, r_{j7}, 2\epsilon r_{j7}) : p \text{ is connected to } R(x_j, r_{j7}) \text{ by an arc of } C_1 \text{ contained in } D(x_j, r_{j7}, 2\epsilon r_{j7})\}$. It is clear that $cl(x_j, r_{j7}) \neq \emptyset$. Two situations are possible:

(i) $cR(x_j, r_{j7}) \neq \emptyset$. In this case

$$i(x_j, r_{j7}) \equiv \inf\{||p - q|| : p \in cl(x_j, r_{j7}),\ q \in cR(x_j, r_{j7})\} = 0.$$

(ii) $cR(x_j, r_{j7}) = \emptyset$.

In fact, if $cR(x_j, r_{j7}) \neq \emptyset$ and $i(x_j, r_{j7}) > 0$, then there are at least four disjoint arcs of C_1 crossing $B(x_j, r_{j_1}) \setminus B(x_j, r_{j7})$, each one of length, at least, $6\epsilon r_j$. Hence

$$H^1(K \cap (B(x_j, r_{j_1}) \setminus B(x_j, r_{j7})) \geq 4 \cdot 6\epsilon r_j = 24\epsilon r_j$$

contradicting again our estimate (3.23). Observe that, in the first case (i), $q(x_j, r_{j7}) \in R(x_j, r_{j7})$ and, in the second one, $q(x_j, r_{j7}) \in l(x_j, r_{j7})$.

Since K_j converges to K as $j \to \infty$, we may choose $j(\epsilon)$ large enough so that

$$d(K_{j(\epsilon)}, K) < \frac{\mu}{3} \tag{3.24}$$

with $\mu < \frac{\epsilon}{2} \inf\{r_j \ : \ j = 1, 2, \cdots, N\}$. Consider a ball $B = B(x_j, r_{j7}) \in F_{17}$. Observe that

$$K_{j(\epsilon)} \cap (B(x_j, r_{jk} - \mu) \setminus D(x_j, r_{jk} - \mu, 2\epsilon r_j + \mu)) = \emptyset \tag{3.25}$$

for $k = 1, 7$. To simplify our notation we write x, r, r_k instead of x_j, r_j, r_{jk} except when it will be convenient to stress the subindex j. Consider a finite set of points $\{p_1, \cdots, p_s\}$ of $C_1 \cap B$, ordered by the arclength parametrization of C_1, such that $|p_i - p_{i+1}| < \frac{\mu}{3}$, $i = 1, 2, \cdots, s-1$. By (3.24), we find points $q_i \in K_{j(\epsilon)}$ in the balls $B(p_i, \frac{\mu}{3})$, $i = 1, 2, \cdots, s$. Observe that $|q_i - q_{i+1}| \le \mu$, $i = 1, 2, \cdots, s-1$. Now, observe that by (3.25), any arc of $K_{j(\epsilon)}$ contained in $B(x, r_7 - \mu)$ exits through $D(x, r_7 - \mu, 2\epsilon r + \mu) \cap \partial B(x, r_7 - \mu)$ (if it exits the ball) which has two connected components which may be called according to their proximity to $l(x, r_7)$, $R(x, r_7)$ by $l(x, r_7, \mu)$, $R(x, r_7, \mu)$, respectively. Let us first suppose that we are in case (i) above. Consider the points of $\{q_1, \cdots, q_s\}$ contained in $D(x, r_7, 2\epsilon r + \mu) \cap B(x, r_7 - \mu)$. Call then $\{m_1, \cdots, m_{s'}\}$, $s' \le s$. If there is an arc of $K_{j(\epsilon)}$ in $D(x, r_7 - \mu, 2\epsilon r + \mu)$ joining (a point of) $l(x, r_7, \mu)$ to a point of $R(x, r_7, \mu)$, then we choose it. Otherwise, no arc of $K_{j(\epsilon)}$ joins $l(x, r_7, \mu)$ to $R(x, r_7, \mu)$. In this case, any arc contained in $K_{j(\epsilon)}$ and passing through some point of $\{m_1, ..., m_{s'}\}$ is connected either to $l(x, r_7, \mu)$ or to $R(x, r_7, \mu)$ but not to both of them. Recall that our purpose is to construct a curve joining $l(x, r_7, \mu)$ to $R(x, r_7, \mu)$ contained, except for a small set, in $K_{j(\epsilon)}$. If there is an arc contained in $K_{j(\epsilon)}$ joining m_1 to $R(x, r_7, \mu)$ then we choose it. Since the distance of this arc to $l(x, r_7, \mu)$ is less than μ, we complete our arc with an artificially added one whose length does not exceed μ. If no arc joining m_1 to $R(x, r_7, \mu)$ exists, then there is an arc in $K_{j(\epsilon)}$ joining m_1 to $l(x, r_7, \mu)$ and we go to the next point m_2 to start the game again. If there is an arc in $K_{j(\epsilon)}$ joining m_2 to $R(x, r_7, \mu)$ then we select such an arc. In this case we have two arcs in $K_{j(\epsilon)}$, one joining m_1 to $l(x, r_7, \mu)$ and the other joining m_2 to $R(x, r_7, \mu)$. We add an artificial segment joining m_1 to m_2 (of length less than μ) to complete a curve joining $l(x, r_7, \mu)$ to $R(x, r_7, \mu)$ and contained in $K_{j(\epsilon)}$ except for a set of length less than μ. If no arc of $K_{j(\epsilon)}$ exists joining m_2 to $R(x, r_7, \mu)$, then there is an arc in $K_{j(\epsilon)}$ joining m_2 to $l(x, r_7, \mu)$ and we go to the next point and start the game again. We continue this strategy until we reach the last point $m_{s'}$. At the end we have a curve, which may have double points, joining $l(x, r_7, \mu)$ to $R(x, r_7, \mu)$ and contained in $K_{j(\epsilon)}$, except for a set of length, at most, μ. Using [Falconer] , Lemma 3.12, we may extract from it a simple curve $\Gamma(x, r_7, \mu)$ contained in $D(x, r_7 - \mu, 2\epsilon r_7 + \mu)$ and contained in $K_{j(\epsilon)}$ except for a set of length, at most, μ. Then, we extend this arc to join the first entrance point of C_1 in $l(x, r_7)$ to the end of $\Gamma(x, r_7, \mu)$ in $l(x, r_7, \mu)$ and the last exit point of C_1 in $R(x, r_7)$ to the end of $\Gamma(x, r_7, \mu)$ in $R(x, r_7, \mu)$. This can be done with a cost in length of, at most, $2(4\epsilon r+\mu)$. Call $\Gamma(x,r_7)$ this extended arc. Then we connect $\Gamma(x,r_7)$ with the previous ball (if $j\geq 2$) $B(x_{j-1},r_{j-17})$ by an arc of C_1 going from $q(x_{j-1},r_{j-17})$ to $p(x_j,r_{j7})$ and with $B(x_{j+1},r_{j+17})$ ($j<N$) by an arc of C_1 going from $q(x_j,r_{j7})$ to $p(x_{j+1},r_{j+17})$.

Let us now consider case (ii). In this case, we take the first and last points of C_1 in $l(x,r_7)$ and we join them by the arc of $l(x,r_7)$ which joins them. The length of this arc does not exceed $4\epsilon r$. As above, we join $q(x_{j-1},r_{j-17})$ to $p(x_j,r_{j7})$ and $q(x_j,r_{j7})$ to $p(x_{j+1},r_{j+17})$ by arcs of C_1.

Since there is only a finite number of balls following the previous specifications, we may construct a curve D^1_ϵ satisfying

$$C_1\subseteq D^1_\epsilon+B(0,2r_0),\qquad D^1_\epsilon\subseteq C_1+B(0,2r_0).$$

Finally, we observe that D^1_ϵ is contained in $K_{j(\epsilon)}$ except for a set of small H^1-measure. In fact, the length of D^1_ϵ not contained in $K_{j(\epsilon)}$ is estimated by:

a) The length of artificial curves used to construct D^1_ϵ inside the balls $B(x_j,r_{j,7})$, $j=1,...,N_{17}$. As we have seen above, for each ball $B(x_j,r_{j7})$ this length is estimated by $2(4\epsilon r_j+\mu)+\mu\leq\frac{19}{2}\epsilon r_j$. Hence, the length contribution of these artificial curves for all balls can be estimated by

$$\sum_{j=1}^{N_{17}}\frac{19}{2}\epsilon r_j\leq\frac{19\epsilon}{4(1-\epsilon)}\sum_{j=1}^{\infty}H^1\big(K\cap B(x_j,r_j)\big)\leq\frac{19\epsilon}{4(1-\epsilon)}H^1(K).\tag{3.26}$$

b) The length of C_1 contained in the balls $B(x_j,r_j)\setminus B(x_j,r_{j7})$, $j=1,...,N_{17}$. Since:

$$\sum_{j=1}^{\infty}H^1\big(K\cap(B(x_j,r_j)\setminus B(x_j,r_{j7}))\big)=H^1(K)-\sum_{j=1}^{\infty}H^1\big(K\cap B(x_j,r_{j7})\big)$$

and, using $(\widetilde{3.17})$, (3.19),

$$\begin{aligned}\sum_{j=1}^{\infty}H^1\big(K\cap B(x_j,r_{j7})\big)&\geq\sum_{j=1}^{\infty}(1-8\epsilon)2r_j\geq\frac{2(1-8\epsilon)}{2+\epsilon}\sum_{j=1}^{\infty}H^1\big(K\cap B(x_j,r_j)\big)\\&=\frac{2-16\epsilon}{2+\epsilon}H^1(K).\end{aligned}$$

Then

$$\sum_{j=1}^{\infty}H^1\big(K\cap(B(x_j,r_j)\setminus B(x_j,r_{j7}))\big)\leq H^1(K)\big(1-\frac{2-16\epsilon}{2+\epsilon}\big)\leq\frac{17}{2}\epsilon H^1(K).\tag{3.27}$$

c) The length of C_1 contained in the balls $B(x_j,r_j)$, $j\geq N+1$, which is estimated by

$$\sum_{j=N+1}^{\infty}H^1\big(K\cap(B(x_j,r_j)\big)\leq\frac{2+\epsilon}{2}\sum_{j=N+1}^{\infty}2r_j<\frac{2+\epsilon}{2}\rho<\frac{2+\epsilon}{2}\epsilon.\tag{3.28}$$

Adding (3.26), (3.27), (3.28) we get

$$H^1\big(D^1_\epsilon \setminus K_{j(\epsilon)}\big) \leq \frac{19\epsilon}{4(1-\epsilon)} H^1(K) + \frac{17}{2}\epsilon H^1(K) + \frac{2+\epsilon}{2}\epsilon.$$

Proceeding in a similar way as we did for D^1_ϵ, we construct curves $D^1_\epsilon, ..., D^p_\epsilon$ such that

$$C_i \subseteq D^i_\epsilon + B(0, 2r_0),\ D^i_\epsilon \subseteq C_i + B(0, 2r_0),$$

$$H^1\big(\bigcup_{i=1}^{p} D^i_\epsilon \setminus K_{j(\epsilon)}\big) \leq \frac{19\epsilon}{4(1-\epsilon)} H^1(K) + \frac{17}{2}\epsilon H^1(K) + \frac{2+\epsilon}{2}\epsilon.$$

The statement of Lemma 3.8 follows by repeating this construction for each n and taking $r_0 = \frac{1}{2n}$, $\epsilon = \frac{1}{n+1}$ at each step. ∎

Proof of Theorem 3.1. As we said in the discussion previous to Lemma 3.8, without loss of generality, we may assume that $c(B_n) = k$ for all n, where $k \geq 1$. We shall give the complete proof only when $k = 1$, the general case being a simple extension of it. Since the ATV of a finite union of parallel segments is zero, by Lemma 3.3 and Lemma 3.6, we may assume that $\sup_n H^1(B_n) < +\infty$. To simplify our presentation let us first consider the case in which $c(B_n) = 1$ for all n, i.e., B_n is a sequence of continua K_n, with $\sup_n H^1(K_n) < +\infty$, converging to a continuum K. We may also assume that B_n is such that $\lim ATV(B_n) = \liminf ATV(B_n)$.

Let ϵ_n be a sequence of positive numbers converging to zero. Define $p(0) = 0$, $C_0 = \emptyset$, $\epsilon_0 = H^1(K)$. Suppose that, at stage n, we have a system of curves $C_0, ..., C_{p(n)}$, such that $C_i \cap C_j = \emptyset$ for $i \neq j$ and

$$H^1(K \setminus (C_0 \cup ... \cup C_{p(n)}) \leq \epsilon_n. \tag{3.29}$$

Then, at stage $n+1$ we extract curves $C_{p(n)+1}, ..., C_{p(n+1)}$ from $K \setminus (C_1 \cup ... \cup C_{p(n)})$ to get a system $C_0, ..., C_{p(n+1)}$ such that $C_i \cap C_j = \emptyset$ for $i \neq j$ and $H^1(K \setminus (C_0 \cup ... \cup C_{p(n+1)}) \leq \epsilon_{n+1}$. Let us consider m fixed. Consider the family of curves $C_1, ..., C_{p(m)}$. Let D^i_n, $i = 1, ..., p(m)$, be as in the statement of Lemma 3.8 satisfying (3.14), (3.15), (3.16) with $p = p(m)$.

Now, let us observe that

$$ATV(K) \leq \sum_{i,j=1}^{p(m)} Inter(C_i, C_j) + 2H^1(K)\,\epsilon_m + \epsilon_m^2. \tag{3.30}$$

Since $\sup_n \sum_{i=1}^{p(m)} H^1(D^i_n) < +\infty$, we may suppose that the arclength parametrization of D^i_n converges to a parametrization of D^i, $i = 1, ..., p(m)$. Hence, they also converges in the Hausdorff distance. Then, for any $\lambda > 0$, letting $n \to \infty$ in (3.14), (3.15) we get

$$D^i \subseteq C_i + B(0,\lambda), \qquad C_i \subseteq D^i + B(0,\lambda).$$

Since C_i, D^i are compact sets and the above inclusions hold for all $\lambda > 0$, we get that $C_i = D^i$. Now, we may apply Lemma 3.7 to get

$$Inter(C_i, C_j) \leq \liminf_n \, Inter(D_n^i, D_n^j). \tag{3.31}$$

For simplicity, write K_n for the sequence K_{j_n} found in Lemma 3.8. Since

$$\begin{aligned} Inter(D_n^i, D_n^j) =& Inter(D_n^i \cap K_n, D_n^j \cap K_n) + Inter(D_n^i \setminus K_n, D_n^j \cap K_n) \\ &+ Inter(D_n^i \cap K_n, D_n^j \setminus K_n) + Inter(D_n^i \setminus K_n, D_n^j \setminus K_n), \end{aligned}$$

it follows that

$$\begin{aligned} Inter(D_n^i, D_n^j) \leq & Inter(D_n^i \cap K_n, D_n^j \cap K_n) + \\ & + 2H^1\left(\cup_{i=1}^{p(m)} D_n^i \setminus K_n\right) M + H^1\left(\cup_{i=1}^{p(m)} D_n^i \setminus K_n\right)^2, \end{aligned} \tag{3.32}$$

where M represents a bound on $H^1(K_n)$ (independent of n). Using (3.30), (3.31), (3.32) and (3.16) , we get

$$\begin{aligned} ATV(K) \leq & \sum_{i,j=1}^{p(m)} \liminf_n \Big(Inter(D_n^i \cap K_n, D_n^j \cap K_n) + \\ & + \frac{36M}{n}(H^1(K)+1) + \frac{324}{n^2}(H^1(K)+1)^2 \Big) + 2H^1(K)\epsilon_m + \epsilon_m^2 \\ \leq & \sum_{i,j=1}^{p(m)} \liminf_n Inter(D_n^i \cap K_n, D_n^j \cap K_n) + 2H^1(K)\epsilon_m + \epsilon_m^2 \\ \leq & \liminf_n \sum_{i,j=1}^{p(m)} Inter(D_n^i \cap K_n, D_n^j \cap K_n) + 2H^1(K)\epsilon_m + \epsilon_m^2 \\ \leq & \liminf_n ATV(K_n) + 2H^1(K)\epsilon_m + \epsilon_m^2. \end{aligned}$$

Since this is true for all m, letting $m \to \infty$ we get

$$ATV(K) \leq \liminf_n ATV(K_n). \tag{3.33}$$

REFERENCES.